CHAPTER 14

CURVES AND SURFACES

14.1 The Conic Sections

PREREQUISITES

1. Recall the concept of an asymptote (Section 3.4).
2. Recall how to sketch a parabola and recognize its equation (Section R.5).

PREREQUISITE QUIZ

1. Find the horizontal and vertical asymptotes of $f(x) = (x + 2)/(x - 3)$.
2. Which are the following equations represent a parabola?

   (a) $y^2 + x^2 = 4y$

   (b) $x = y^2 - 3$

   (c) $-x^2 - 2 = y$
3. Sketch the graph of any parabolas whose equation is given in Question 2.

GOALS

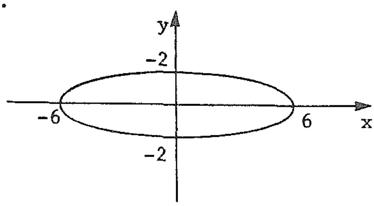
1. Be able to write equations of ellipses, parabolas, and hyperbolas in standard form and sketch them.

STUDY HINTS

1. Ellipse. The standard form is $x^2/a^2 + y^2/b^2 = 1$. In an ellipse, the sum of the distances from two foci to a point on the ellipse is constant. You often don't need to be able to find the foci, but you should be able to convert to the standard form and use the equation to sketch an ellipse. The x-intercepts are $\pm a$ and the y-intercepts are $\pm b$, found by substituting $y = 0$ and $x = 0$, respectively.
2. Hyperbola. The standard form is either $x^2/a^2 - y^2/b^2 = 1$ or $y^2/a^2 - x^2/b^2 = 1$. Here, the difference of the distances from two foci to a point on the hyperbola is constant. Again, you probably don't always need to find the foci, but you should be able to convert to the standard form and sketch a hyperbola. By drawing a rectangle with sides at $x = \pm a$ and $y = \pm b$, the diagonals of the rectangle represent the asymptotes $y = \pm(b/a)x$. The intercepts are either $x = \pm a$ or $y = \pm b$. Substitution determines which are the intercepts.
3. Circles. A circle is a special ellipse in which $a = b$.
4. Parabolas. Your main concern is to be able to sketch a parabola. It is less important to be able to determine the directrix and the focus.
5. Circles and parabolas. Look back over Section R.5 and make sure you understand that material in the present context.

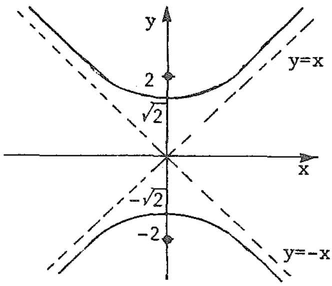
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



Dividing through by 36, the equation becomes $x^2/(6)^2 + y^2/(2)^2 = 1$. Since $2 < 6$, the foci are $(\pm c,0)$, where $c = \sqrt{6^2 - 2^2}$, i.e., the foci are $(\pm\sqrt{32},0)$. The intercepts are $(\pm 6,0)$ and $(0,\pm 2)$.

5.



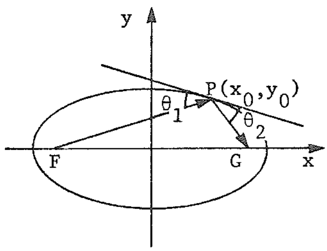
Dividing through by 4, $y^2 - x^2 = 2$ is equivalent to $y^2/(\sqrt{2})^2 - x^2/(\sqrt{2})^2 = 1$. Thus, the foci are $(0,\pm c)$, where $c = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$, i.e., the foci are $(0,\pm 2)$. The asymptotes are $y = \pm(\sqrt{2}/\sqrt{2})x = \pm x$, and the intercepts are at $(0,\pm\sqrt{2})$.

9. From the given information, we have $c = 4$ and $a = 1/4c = 1/16$. Therefore, the equation of the parabola is $y = x^2/16$.

13. The equation has the form $x = by^2$, so the focus is at $(c,0)$, where $c = 1/4b = 1/4$. Thus, the focus is $(1/4,0)$ and the directrix is $x = -1/4$.

17. Since the vertex is at $(0,0)$, try the general form $y = ax^2$. Since the parabola passes through $(2,1)$, $y = ax^2$ becomes $1 = 4a$. Thus, $a = 1/4$ and the equation is $y = x^2/4$.

21. Use the method of Example 6. The equation of the parabola is $y = ax^2$. Since $y = 0.3$ when $x = 0.4$, we get $a = 0.3/0.16 = 15/8$. The focus is at $(0,c)$, where $a = 1/4c$, so $c = 1/4a = 1/(15/2) = 2/15$. Thus, the light source should be placed on the axis, 2/15 meters from the mirror.

25. 

Consider the equation of the ellipse in standard form with $b < a$: $x^2/a^2 + y^2/b^2 = 1$. Let P be on the ellipse at (x_0,y_0) . Implicit differentiation yields $2x/a^2 + (2y/b^2)(dy/dx) = 0$. Therefore, the slope of the tangent line at P is $dy/dx = -b^2x_0/a^2y_0$. Since the foci are located at $(\pm c,0)$, we have $\overrightarrow{FP} = (x_0 + c)\underline{i} + y_0\underline{j}$ and $\overrightarrow{PG} = (c - x_0)\underline{i} - y_0\underline{j}$. Also, the direction of the tangent line is $\vec{d} = -a^2y_0\underline{i} + b^2x_0\underline{j}$. Now, by the definition of the dot product, $\vec{d}\cdot\overrightarrow{FP} = (\cos\theta_1)\|\vec{d}\|\|\overrightarrow{FP}\| = -a^2y_0(x_0 + c) + b^2x_0y_0 = (b^2 - a^2)x_0y_0 - a^2y_0c = -(c^2x_0y_0 + a^2y_0c) = -cy_0(cx_0 + a^2)$ and $\vec{d}\cdot\overrightarrow{PG} = (\cos\theta_2)\|\vec{d}\|\|\overrightarrow{PG}\| = a^2y_0(c - x_0) + b^2x_0y_0 = (b^2 - a^2)x_0y_0 + a^2y_0c = -(c^2x_0y_0 - a^2y_0c) = -cy_0(cx_0 - a^2)$. We used the fact that $c^2 = a^2 - b^2$. Rearrangement gives us $\cos\theta_1 = -cy_0(cx_0 + a^2)/\|\vec{d}\|\|\overrightarrow{FP}\|$ and $\cos\theta_2 = -cy_0(cx_0 - a^2)/\|\vec{d}\|\|\overrightarrow{PG}\|$.

From the figure, if the angle from $\vec{d}$ to $\overrightarrow{PG}$ is $-\phi$, then the angle from $\vec{d}$ to $\overrightarrow{FP}$ is $\pi - \phi$. Thus, we need to show that $\cos\theta_1/\cos\theta_2 = -1 = [-cy_0(cx_0 + a^2)/\|\vec{d}\|\|\overrightarrow{FP}\|]/[-cy_0(cx_0 - a^2)/\|\vec{d}\|\|\overrightarrow{PG}\|]$. This simplifies to $(cx_0 + a^2)/\|\overrightarrow{FP}\| = -(cx_0 - a^2)/\|\overrightarrow{PG}\|$. Use the fact that $\|\overrightarrow{FP}\| = \sqrt{(x_0 + c)^2 + y_0^2}$ and $\|\overrightarrow{PG}\| = \sqrt{(c - x_0)^2 + y_0^2}$.

Rearrange the equation and square to get $\|\overrightarrow{PG}\|^2(cx_0 + a^2)^2 = \|\overrightarrow{FP}\|^2(cx_0 - a^2)^2$ or $((c - x_0)^2 + y_0^2)(cx_0 + a^2)^2 = ((x_0 + c)^2 + y_0^2) \times (cx_0 - a^2)^2$. Multiplying out, cancelling like terms, dividing by four, and rearrangement yields $-cx_0^3a^2 - c^3x_0a^2 - cx_0a^2y_0 = c^3x_0^3 + cx_0a^4$, i.e., $-cx_0a^2(x_0^2 + c^2 + y_0^2) = cx_0(c^2x_0^2 + a^4)$, i.e., $-a^2(x_0^2 + c^2 + y_0^2) = c^2x_0^2 + a^4$. Now, $x_0^2/a^2 + y_0^2/b^2 = 1$ rearranges to $y_0^2 = b^2 - x_0^2b^2/a^2$ and using $c^2 = a^2 - b^2$ makes the left-hand side $-a^2(x_0^2 + c^2 + y_0^2) =$

25. (continued)

$$-a^2x_0^2 - a^2c^2 - a^2b^2 + x_0b^2 = x_0^2(b^2 - a^2) - a^2(c^2 + b^2) = -c^2x_0^2 - a^4 ,$$

which is the same as the right-hand side except for an irrelevant sign change.

SECTION QUIZ

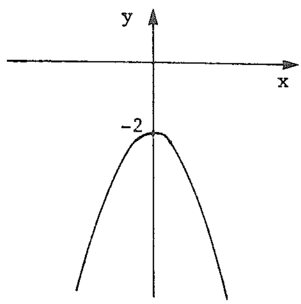
1. Classify the equations of the following conics:
   (a) $x^2 = 4(1 - y^2)$
   (b) $5y^2 - x = 0$
   (c) $4x^2 - y^2 = -4$
   (d) $4x^2 = 1 - 4y^2$
2. Which of the equations in Question 1 are functions of x ?
3. Sketch the conic sections described by the equations in Question 1.
4. What is the equation of the ellipse passing through the points $(-5,0)$, $(0,-7)$, $(5,0)$, and $(0,7)$?
5. Mother and father racoon were raiding separate campgrounds, looking for food to feed their family. Being expert food thieves, the two racoons had their methods synchronized. As they were breaking into separate backpacks, a hunter in the distance sighted a duck. A shot sent both racoons scurrying back into the wilderness without any food.
   (a) Suppose the campgrounds are separated by 900 m and the sound travels at 300 m/sec . If one racoon heard the shot 1 second before the other, describe all points where the hunter could possibly have been. [Hint: Let the campsites represent foci located at $(\pm 450,0)$.]
   (b) Sketch the graph of the expression found in (a).

ANSWERS TO PREREQUISITE QUIZ

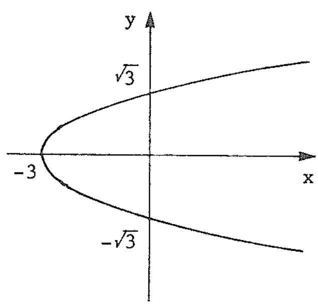
1. Horizontal asymptote: $y = 1$; vertical asymptote: $x = 3$

2. b and c

3. (b)



(c)



ANSWERS TO SECTION QUIZ

1. (a) Ellipse

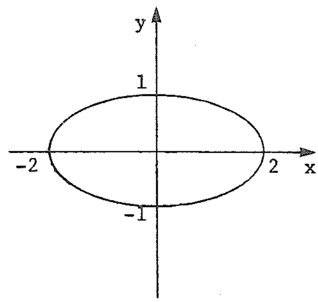
   (b) Parabola

   (c) Hyperbola

   (d) Circle

2. None of them

3. (a)



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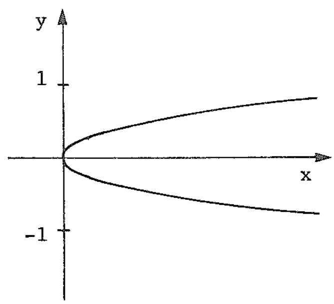
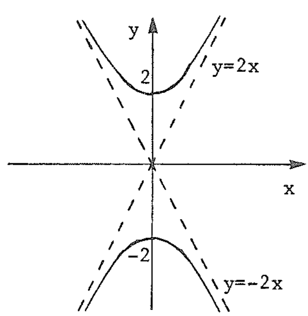
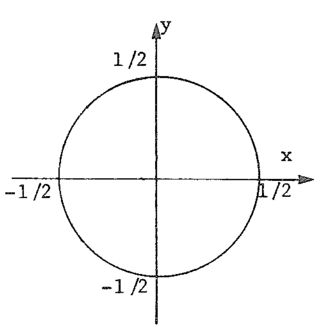


(c)



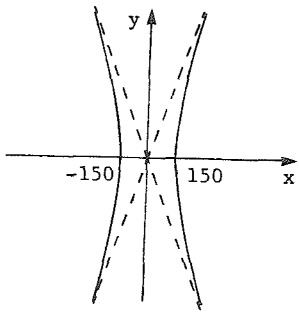
(d)



4. $x^2/25 + y^2/49 = 1$

5. (a) $x^2/22500 - y^2/180000 = 1$

(b)



14.2 Translation and Rotation of Axes

PREREQUISITES

1. Recall how to find equations of shifted parabolas (Section R.5).
2. Recall the general equations for the conic sections (Sections R.5 and 14.1).

PREREQUISITE QUIZ

1. What kind of conic section is represented by each of the following equations?
   (a) $x^2 + y^2 = 4$
   (b) $2x^2 + 3y^2 = 6$
   (c) $y^2 - x^2 = 2$
   (d) $y = x^2 + 2$
2. What is the vertex of the parabola with equation $y^2 + 2y + 1 = x - 1$?

GOALS

1. Be able to translate and rotate axes for the purpose of graphing conic sections.

STUDY HINTS

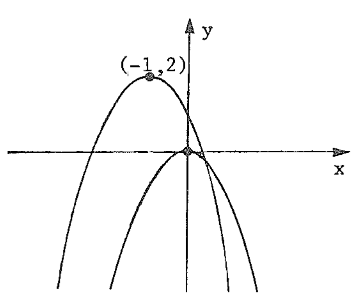
1. Translating axes. Translation of axes is called for if linear terms in x and y appear, but no cross-term xy appears. For purposes of sketching, let $X = x - p$ and $Y = y - q$. This allows you to write equations in standard form. Essentially, all that is being done is a shifting of the origin. It is a good idea to substitute a point into the original equation to determine if the origin was shifted in the correct direction.

2. Rotating axes. If the term xy is present, rotating axes can get rid of the term. The equations used to change (x,y) to (X,Y) are given in formula (5), p. 705. You should either remember these formulas or learn to derive them. To convert back to (x,y) from (X,Y), use the angle $-\alpha$. Although these formulas should be known, it is a good idea to consult your instructor to see if you will be tested on them.

3. General conics. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is the general equation of a conic. By computing $B^2 - 4AC$, you should know that the graph (provided it exists) is an ellipse if this quantity is less than zero, is a parabola if $B^2 - 4AC = 0$, and is a hyperbola if $B^2 - 4AC > 0$. An important formula is $\tan 2\alpha = B/(A - C)$; this determines the angle of rotation, α. Finally, substituting the expressions in (5) helps you to get the standard form.

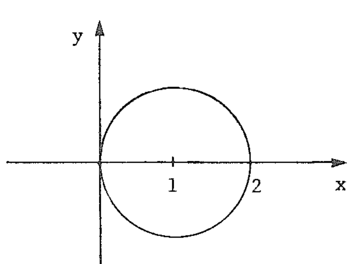
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.

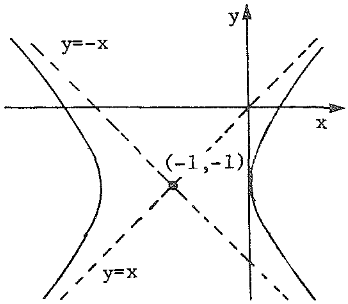


$y = -x^2$ is a parabola opening downward and centered along the y-axis. Its vertex is $(0,0)$. $y - 2 = -(x + 1)^2$ is the same parabola with the origin shifted to $(-1,2)$.

5.



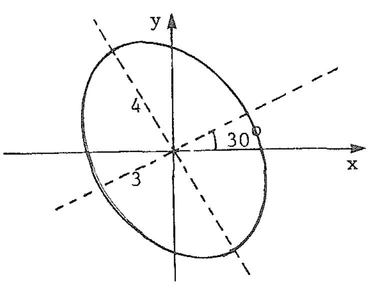
Complete the squares. $(x^2 - 2x + 1) + y^2 = 1 = (x - 1)^2 + y^2$ is a circle of radius 1 centered at $(1,0)$.

9. 

By completing the squares, the equation becomes $(x^2 + 2x + 1) - (y^2 + 2y + 1) = 1 + 1 - 1 = 1 = (x + 1)^2 - (y + 1)^2$. This has the form $X^2 - Y^2 = 1$, which is a hyperbola with asymptotes $Y = \pm X$ and intercepts $(\pm 1,0)$. The desired hyperbola has its origin shifted to $(-1,-1)$.

13. Use the method of Example 4. $\cos 15^\circ \approx 0.97$ and $\sin 15^\circ \approx 0.26$, so the transformation of coordinates is $x = 0.97X - 0.26Y$, $y = 0.26X + 0.97Y$, and $X = 0.97x + 0.26y$, $Y = -0.26x + 0.97y$.

17. We consider the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. It is an ellipse if $B^2 - 4AC < 0$, a hyperbola if $B^2 - 4AC > 0$, and a parabola if $B^2 - 4AC = 0$. In this case, $A = 19/4$, $B = 7\sqrt{3}/6$, and $C = 43/12$, so $B^2 - 4AC = 49/12 - 4(19/4)(43/12) = (49 - 19 \cdot 43)/12 < 0$. Thus, it is an ellipse.

21. 

Follow the four-step procedure described in the box on p. 709. $\alpha = (1/2)\tan^{-1}(B/(A - C)) = (1/2)\tan^{-1}[(7\sqrt{3}/6)/(19/4 - 43/12)] = (1/2)\tan^{-1}(\sqrt{3}) = \pi/6$. Thus, $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$, and $x = \sqrt{3}X/2 - Y/2$ and $y = X/2 + \sqrt{3}Y/2$; therefore, the given equation is $(19/4)(3X^2/4 - \sqrt{3}XY/2 + Y^2/4) + (43/12)(X^2/4 + \sqrt{3}XY/2 + 3Y^2/4) + (7\sqrt{3}/6)(\sqrt{3}X^2/4 + XY/2 - \sqrt{3}Y^2/4) - 48 = 16X^2/3 + 3Y^2 - 48 = 0$, i.e., $X^2/9 + Y^2/16 = 1$. This new equation is an ellipse with intercepts $(\pm 3,0)$ and $(0,\pm 4)$. Rotate this by $\pi/6$ to get the desired sketch.

25. Since the vertex is at $(1,0)$, try the equation $y = a(x - 1)^2$. The parabola passes through $(2,1)$, so the equation becomes $1 = a$. As a check, $(0,1)$ also satisfies $y = (x - 1)^2$.

29. Here, $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$. Thus, $x = X/2 - \sqrt{3}Y/2$ and $y = \sqrt{3}X/2 + Y/2$. Substitution gives $x^2 + 2x + y^2 - 2y = (X^2/4 - \sqrt{3}XY/2 + 3Y^2/4) + (X + \sqrt{3}Y) + (3X^2/4 + \sqrt{3}XY/2 + Y^2/4) - (\sqrt{3}X + Y) = X^2 + Y^2 + (1 - \sqrt{3})X + (\sqrt{3} - 1)Y = 2$.

33. Since $B^2 - 4AC < 0$, the equation $Ax^2 + Bxy + Cy^2 = 1$ represents an ellipse. From formulas (9), this becomes $\bar{A}X^2 + \bar{C}Y^2 = 1$, where $A = \bar{A}\cos^2\alpha + \bar{C}\sin^2\alpha$, $C = \bar{A}\sin^2\alpha + \bar{C}\cos^2\alpha$, and $\tan 2\alpha = B/(A - C)$. $\bar{A}X^2 + \bar{C}Y^2 = 1$ represents an ellipse whose axes have length $1/\sqrt{\bar{A}}$ and $1/\sqrt{\bar{C}}$. Therefore, the area is $\pi/\sqrt{\bar{A}\bar{C}}$. But $AC - B^2/4 = \bar{A}\bar{C}$ from p. 707 and this gives the desired result.

SECTION QUIZ

1. What type of conics are described by the following equations?

   (a) $3u^2 - 5uv - 2v^2 + 8v - 3u = 0$

   (b) $y^2 + 4x^2 - 8x + 4y - 17 = 3$

   (c) $t^2 + 4xt + 4x^2 - 3t + 4x = 28$

   (d) $4x + xy - 2y^2 = 1$

2. Which of the equations in Question 1 represent rotated conics?

3. A circle of radius 3 is centered at $(-2,1)$ and it is rotated $30°$. What is the equation of the circle?

4. Sketch the graph of $(y^2 - x^2)/2 + \sqrt{3}xy = 0$.

5. Sketch the graph of the equation in Question 1(b).

6. Stranded in the middle of the Pacific Ocean on a rubber life raft, an unlucky gentleman has received the company of a shark. While worrying that the shark's sharp teeth will soon puncture the raft, our sea-faring friend notices that the shark is swimming in an elliptical path. He notes that if he is at the origin, the major axis is on the line $y = \sqrt{3}x$. The major axis has length 3 and the minor axis, whose length is 2 , intersects the major axis at $(1,\sqrt{3})$.

   (a) What is the equation of the ellipse?

   (b) Will the shark collide with the man? Explain.

ANSWERS TO PREREQUISITE QUIZ

1. (a) Circle

   (b) Ellipse

   (c) Hyperbola

   (d) Parabola

2. $(1,-1)$

ANSWERS TO SECTION QUIZ

1. (a) Hyperbola

   (b) Ellipse

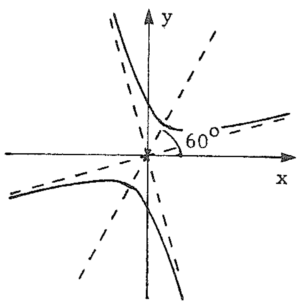
   (c) Parabola

   (d) Hyperbola

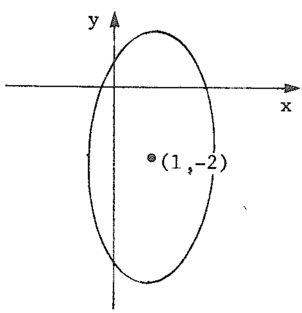
2. a , c , and d

3. $x^2 + 4x + y^2 + 2y = 4$

4.



5.



6. (a) $85x^2 - 10\sqrt{3}xy + 21y^2 - 4x - 20\sqrt{3}y = 20$

(b) No; (0,0) does not satisfy the equation in (a).

14.3 Functions, Graphs, and Level Surfaces

PREREQUISITES

1. Recall the concept of functions in one variable (Section R.6).
2. Recall how to graph functions of one variable (Section 3.4).

PREREQUISITE QUIZ

1. What is the domain of the following functions?
   (a) $f(t) = 2/(t - 3)$
   (b) $g(x) = \sqrt{x^2 - 3x + 2}$
2. Sketch the graphs of the functions in Question 1.

GOALS

1. Be able to define a function of several variables, a level curve, and a level surface.
2. Be able to recognize the equations of a plane or sphere and sketch the graphs.
3. Be able to apply the method of sections to sketch surfaces in space.

STUDY HINTS

1. Function defined. Recall that a function of one variable assigns a single, real value for each x in its domain. A function of two variables assigns a single, real value for each (x,y) in its domain. Similarly, a function of three variables assigns a single, real value for each (x,y,z) in its domain.
2. Level curves and surfaces. A level curve is the intersection of $f(x,y)$ and the plane $z = c$. You just sketch $f(x,y) = c$ in the xy-plane. A level surface is the surface which results if $f(x,y,z)$ is assigned a constant value. This concept is useful for sketching in three dimensions.

3. Sketching planes. Many of us are poor artists, and as a result, three-dimensional geometry may be frustrating due to this problem rather than a lack of mathematical understanding. Planes are most easily sketched by plotting three non-collinear points (usually on the coordinate axes) and then passing a plane through them.

   Another simple method is to set one of x , y , or z constant to obtain lines as level curves. Then use the method of sections described in Study Hint 6.

4. Spheres. Example 5(b) shows a surface of the form $x^2 + y^2 + z^2 = c^2$. This is a sphere of radius c centered at the origin. More generally, $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ is a sphere of radius r centered at (a,b,c) .

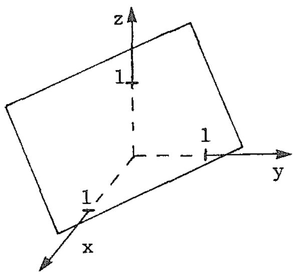
5. Cylinders. A surface is called a cylinder if x , y , or z is missing from its equation. A cylinder can be sketched by drawing the level curve in the plane where the missing variable is zero. Then move the curve along the axis of the missing variable, since no restrictions are placed upon the missing variable. Note that, in Example 6(b), the level curve was drawn in the plane $z = 0$ and then moved along the z-axis since z is missing in the equation. Cylinders may be classified according to their cross-sections as being elliptic, parabolic, circular, or hyperbolic.

6. Method of sections. The idea is to sketch level curves in planes parallel to the coordinate axes. Then, we put them together to obtain the three dimensional figure. If possible, try to look for a general pattern. For instance, in Example 7, we note that in the plane $z = c$, the level curve is a circle of radius c is $z \geqslant 0$. Sometimes, these level curves are called traces.

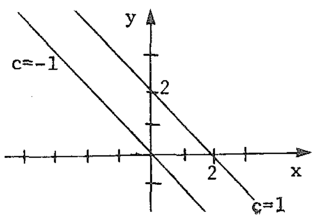
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The domain is (x,y) such that the denominator does not vanish. $x = 0$ is the y-axis. The domain is all points not on the y-axis; $f(1,0) = 0$; $f(1,1) = 1$.

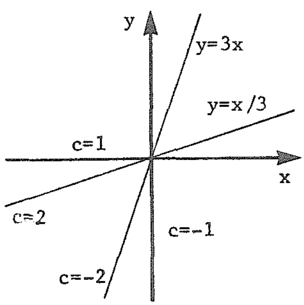
5. The domain is (x,y) such that the denominator does not vanish. $x^2 + y^2 + z^2 = 1$ is the unit sphere. The domain is all points not on the unit sphere; $f(1,1,1) = 1$; $f(0,0,2) = -2/3$.

9. 

$f(x,y) = 1 - x - y$ represents a plane. The plane contains $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$. Plot the three noncollinear points and sketch the plane which passes through all three.

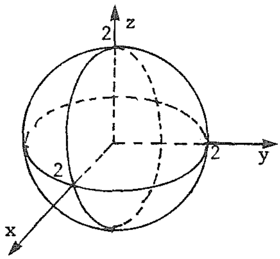
13. 

$1 - x - y = 1$ implies $x + y = 0$, and $1 - x - y = -1$ implies $x + y = 2$. Thus, the level curves are straight lines.

17. 

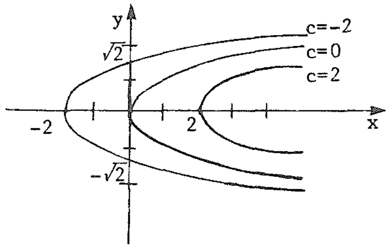
$c = (x + y)/(x - y)$ implies $cx - cy = x + y$, i.e., $(c - 1)x = (c + 1)y$ or $y = (c - 1)x/(c + 1)$. The general level curve is a line with slope $(c - 1)/(c + 1)$.

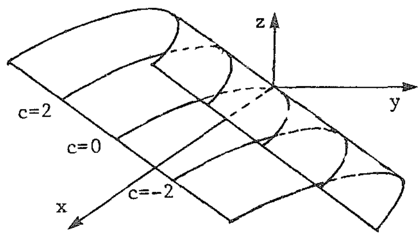
21.



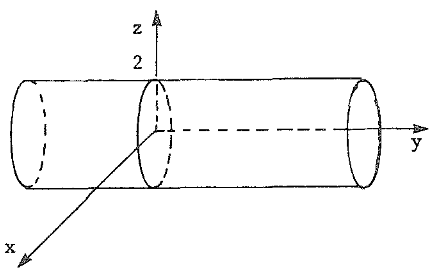
This is a sphere centered at the origin with radius 2 .

25. $c = x - y^2$ implies $y^2 = x - c$. The level curves are parabolas centered on the x-axis. (See below, left.) Its vertex is located at $(c,0)$ and the parabola opens up in the positive x-direction. Putting these curves together gives us a parabolic cylinder with the individual vertices located on the line $x = z$ in the xz-plane (below, right).



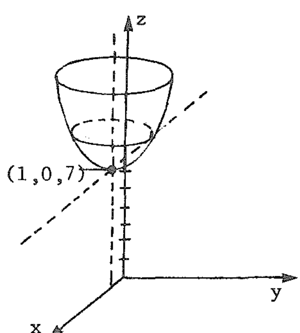


29.



Since the variable y is missing, $z^2 + x^2 = 4$ represents a cylinder along the y-axis. On the xz-plane, $z^2 + x^2 = 4$ is a circle centered at the origin and with radius 2 .

33.

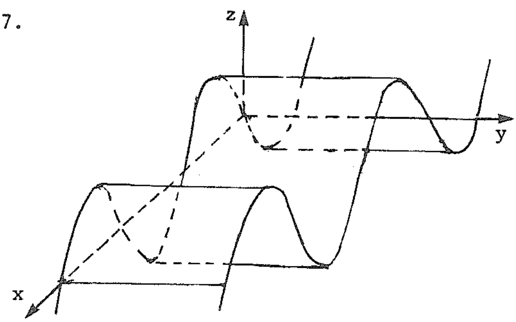


Completing the square yields $z = (x^2 - 2x + 1) + y^2 + 7$ or $z - 7 = (x - 1)^2 + y^2$, which is the same surface as in Exercise 31 except that this surface is shifted up by 7 units. When z is constant, the equation describes a circle centered at $(1,0)$ with radius $\sqrt{z - 7}$.

33. (continued)

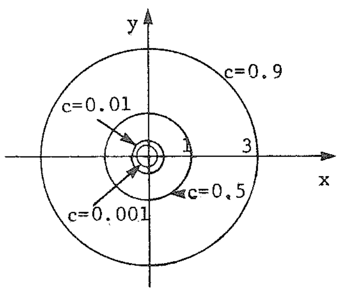
Putting these level curves together gives us a paraboloid as in Example 7. z must be greater than or equal to 7 .

37.



The y variable is missing from the equation, so the graph is a "cylinder". In the xz-plane, sketch the curve $z = \sin x$. Then move the curve along the y-axis in both directions.

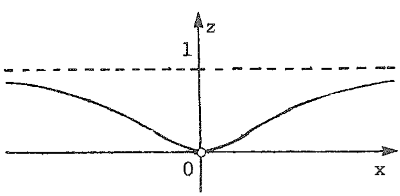
41. (a)



$c = e^{-1/(x^2+y^2)}$ implies $\ln c = -1/(x^2 + y^2)$ or $-1/\ln c = x^2 + y^2$. This is a circle centered at the origin with radius $\sqrt{-1/\ln c}$ if $0 < c < 1$.

(b) An exponential of any number is positive, so the level curves of $c < 0$ do not exist. $x^2 + y^2$ is a positive quantity, so $-1/(x^2 + y^2) < 0$; therefore $e^{-1/(x^2+y^2)} < 1$, so no level curves exist for $c > 1$.

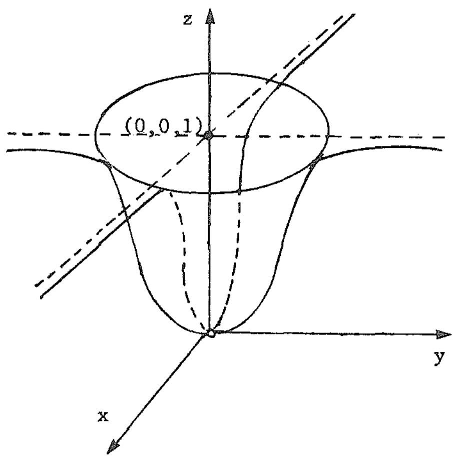
(c)



If $y = 0$, then $f(x,y) = z = e^{-1/x^2}$. $z'(x) = (-2/x^3)e^{-1/x^2}$, so there are no critical points. The graph is symmetric about the z-axis. The domain is $x \neq 0$. The exponent, $-1/x^2$, is always negative, so the range is $0 < z < 1$. $\lim_{x\to 0} z = 0$ and $\lim_{x\to\infty} z = 1$, so $z = 1$ is an asymptote.

41. (d) In polar coordinates, the equation is $f(r,\theta) = e^{-1/r^2}$. This equation is independent of θ , so the cross-section of any vertical plane through the origin looks the same.

(e)



The surface consists of a series of circles placed on top of each other. The radii of the circles increase exponentially. Putting these together gives us a bell-shaped figure which asymptotically approaches the plane $z = 1$.

SECTION QUIZ

1. Classify the following as the equation of a sphere, a plane, or a cylinder:

(a) $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 4$

(b) $(x - 3) + (y + 2) + (z - 1) = 4$

(c) $2x - 5y + z = 0$

(d) $(x - 3)^2 + (y + 2) = 4$

(e) $y^2 - x^2 - 2 = 0$

2. Sketch the surfaces in Question 1.

3. (a) Define a level curve.

(b) Describe the level curves of $z = f(x,y)$ if the function's graph is a plane.

(c) Sketch the level curves for the function in Question 1(e) if $c = -2, -1, 0, 1, 2$.

4. Sketch the graph of $f(x,y) = x^3 - x$.

5. Jealous Johnny saw Phil the Flirt dancing with his girl friend last night. The revenge-minded Johnny dreamed up a devious plan. First, he was going to trap Phil the Flirt inside a structure described by the equation $(x - 1)^2 + (y + 1)^2 + (z - 2)^2 = 1$. Openings at $z = 2$ permitted feathers to enter each hour to tickle Phil. Finally, if Phil should try to escape, he would simply fall onto a smooth slide described by $3x + y - z = 6$ and slide into a huge pool of anchovies.

   (a) Describe the level curve for $f(x,y) = 2$ where the feathers entered.

   (b) Sketch the structure which would entrap Phil the Flirt.

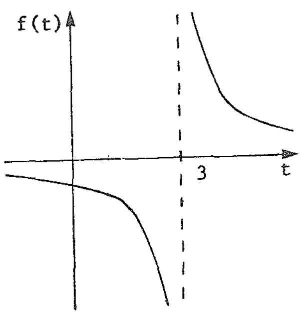
   (c) Sketch the slide leading into the pool of anchovies.

ANSWERS TO PREREQUISITE QUIZ

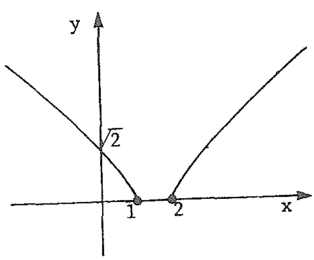
1. (a) $t \neq 3$

   (b) $x \leqslant 1$ and $x \geqslant 2$

2. (a)



   (b)



ANSWERS TO SECTION QUIZ

1. (a) Sphere

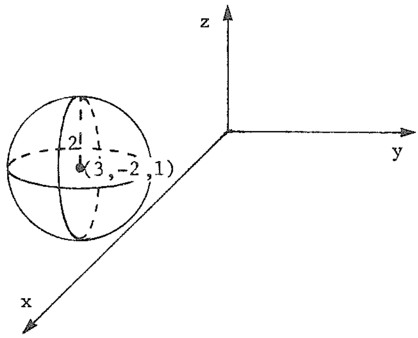
   (b) Plane

   (c) Plane

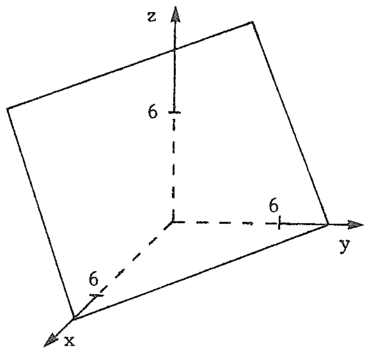
   (d) Cylinder

   (e) Cylinder

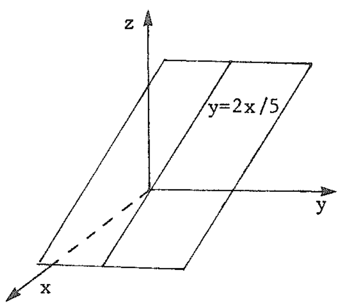
2. (a)



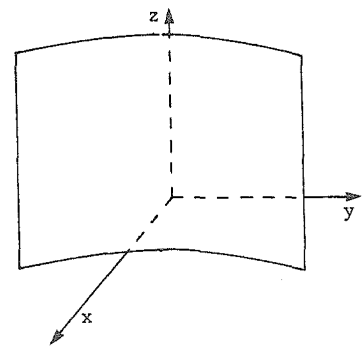
(b)



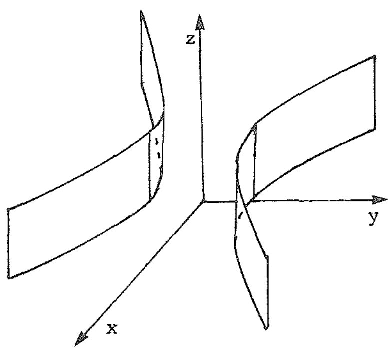
(c)



2. (d)



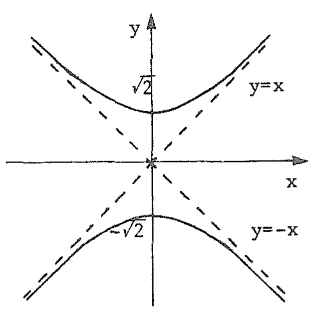
(e)



3. (a) A level curve is the set of all (x,y) in the plane satisfying $f(x,y) = c$, where c is a constant.

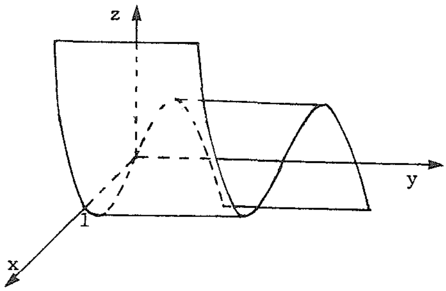
(b) The level curves are straight lines.

(c)



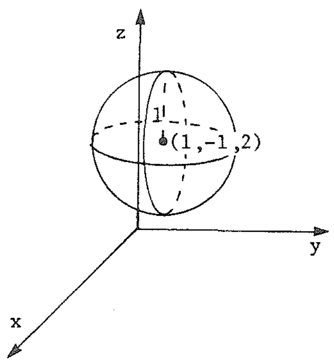
All of these level curves are the same.

4.

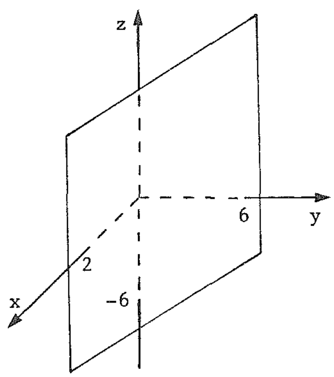


5. (a) It is a circle of radius 1 centered at (1,-1) .

(b)



(c)



14.4 Quadric Surfaces

PREREQUISITES

1. Recall how to sketch level curves (Section 14.3).
2. Recall how to recognize equations of cylinders and sketch them (Section 14.3).

PREREQUISITE QUIZ

1. Let $z = x^2 + y^2$. Sketch the level curves in the xy-plane for $z = 0$, 1 , and 3 .
2. Sketch the graph of $x^2 + 4y^2 = 4$ in space.
3. Describe how to recognize the equation of a cylinder.

GOALS

1. Be able to sketch quadric surfaces.

STUDY HINTS

1. Quadric surfaces. The surfaces in this section may all be sketched by using the method of sections, which was discussed in Section 14.4. Quadric surfaces may be classified by the form of the equation (see Study Hints 2,4,5,6,7, and 8). If the variables are interchanged, you should be able to analyze the equations in a manner analogous to the text's discussion. The difference is that these surfaces will be "lying on their sides" rather than "standing up." For example, $z = 4x^2 - 3y^2$ and $y = 4z^2 - 3x^2$ are both hyperbolic paraboloids.

2. Hyperbolic paraboloid. The general equation is $z = ax^2 - by^2$, where a and b have the same sign. The level curves for constant z are hyperbolas, except when $z = 0$. The hyperbolas are centered about a different axis when z changes sign. The hyperbolas degenerate to two straight lines when $z = 0$. See Example 1 and Exercise 17.
3. Saddle points. At these points, the surfaces are rising in certain directions and falling in others. For example, in Fig. 14.4.2, the function has a local minimum if we set $y = 0$. If $x = 0$, we have a local maximum. More details are discussed in Chapter 16.
4. Hyperboloid of two sheets. The general equation is $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$. In each plane parallel to the xy-plane, the level curve is an ellipse with equation $x^2/a^2 + y^2/b^2 = -1 + z^2/c^2$. Thus, no level curve exists if $z^2/c^2 < 1$. It is easier to figure out this fact rather than memorizing it. For example, at $z = 0$, there are no points, so there must be a "gap." Note that in planes parallel to the xz- and yz-planes, the level curve is a hyperbola. See Example 4.
5. Ellipsoid. The general equation is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. These surfaces are "footballs" with intercepts at $x = \pm a$, $y = \pm b$, $z = \pm c$. Any level curve is an ellipse which gets smaller as you move away from the planes defined by the coordinate axes, i.e., the xy-, xz-, or yz-planes, to the intercepts. See Example 5.
6. Hyperboloid of one sheet. The general equation is $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$. Note the similarities and differences with the two-sheeted hyperboloid. In the general form, one equation's right-hand side equals 1 and the other has -1. Parallel to the xy-plane, both are ellipses for constant z. However, ellipses exist for all z in the one-sheet form. See Example 6.

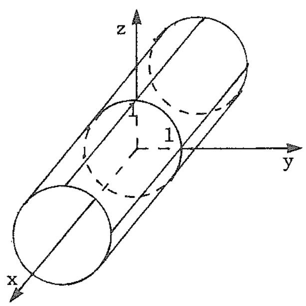
7. Elliptic paraboloid. The general form is $z = ax^2 + by^2$, where a and b have the same sign. If z is constant, the level curves are ellipses. The level curves are smallest when $z = 0$ (a point) and the level curves get larger with $|z|$. Level curves in planes parallel to the xz- and yz-planes are parabolas. These surfaces are bowl shaped. See Exercise 18.

8. Elliptic cone. The general form is $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$. By rearranging, we get $z^2 = c^2x^2/a^2 + c^2y^2/b^2$. As with the elliptic paraboloid, the level curves for constant z are again ellipses. The general shape, as the name suggests, is a cone. See Exercise 25.

9. Origin of conics. Examples 8 and 9 prove that slices of a cone are conics. Except in honors classes, you will probably not have to reproduce similar arguments.

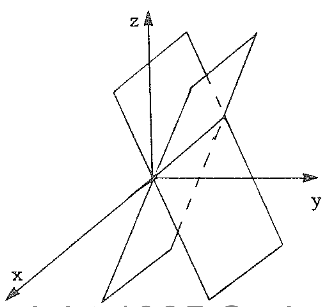
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



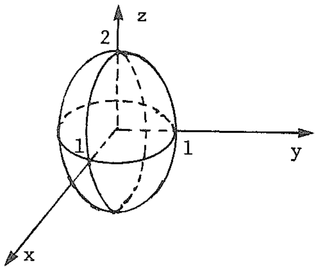
$y^2 + z^2 = 1$ represents a unit circular cylinder along the x-axis.

5.



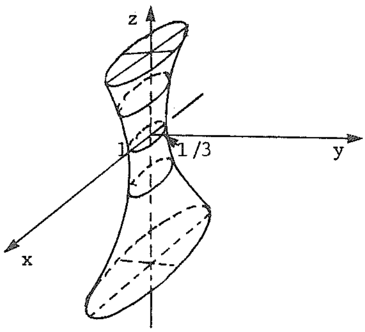
$z^2 - 8y^2 = 0$ if $z = \pm\sqrt{8}y$. Thus, in three dimensions, this represents two intersecting planes. The planes intersect on the x-axis.

9.



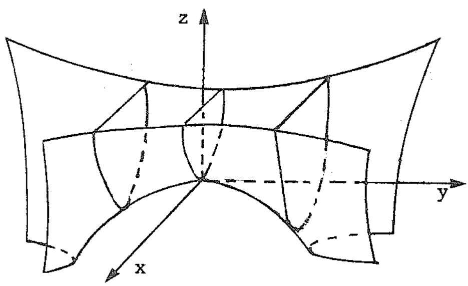
$x^2 + y^2 + z^2/2 = 1$ is an ellipsoid (see Example 5). The intercepts are $x = \pm 1$, $y = \pm 1$, and $z = \pm 2$.

13.



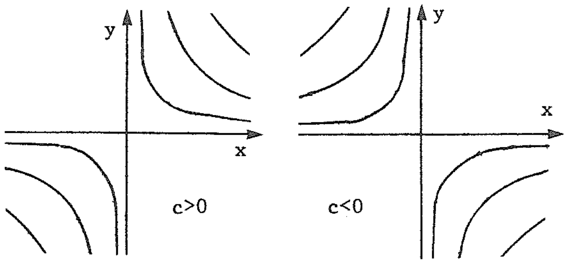
Rearrangement yields $x^2 + 9y^2 = z^2 + 1$ which is $x^2 + 9y^2 = 1$ in the xy-plane. This is an ellipse with x-intercepts, ± 1, and y-intercepts, $\pm 1/3$. Note that the graph is symmetric to the xy-plane. As z approaches $\pm\infty$, the cross-sections are ellipses of increasing size. The surface is a hyperboloid of one sheet. (See Example 6.)

17. (a)



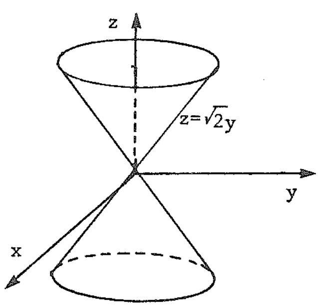
The level curve for $c = 0$ is $x^2 = 2y^2$ or $x = \pm\sqrt{2}y$, which is two straight lines. For $c > 0$, $c = x^2 - 2y^2$ describes a hyperbola. Rewriting this, we get $1 = x^2/c - 2y^2/c$, so the hyperbola is bounded by the lines $x = \pm\sqrt{2}y$. As c increases, the vertices of the hyperbola move further from the z-axis. For $c > 0$, the hyperbola is centered around the x-axis. If $c < 0$, then the hyperbola is still bounded by the lines $x = \pm\sqrt{2}y$, but these hyperbola are centered around the y-axis. For decreasing c, the hyperbola moves further from the z-axis.

17. (b)



When $c = 0$, the level curve is $x = 0$ and $y = 0$, two straight lines. For $c > 0$, the hyperbola exists in the first and third quadrants. For $c < 0$, the hyperbola lies in the second and fourth quadrants. For $c = 2, 1, 0, -1, -2$, the level curves are the same as those in Fig. 14.4.1, except that the curves are rotated 45° to the left. Thus, the surface is also rotated 45°, so $z = xy$ determines a hyperbolic paraboloid.

21.



This cone is composed of cross-sections which are ellipses. For constant z, the ellipse is $1 = x^2/z^2 + y^2/(z^2/2)$.

25. (a) The equation can be rewritten as $x^2/a^2 + y^2/b^2 = z^2/c^2$ or $c^2x^2/a^2 + c^2y^2/b^2 = z^2$. For a constant z, we have $c^2x^2/a^2z^2 + c^2y^2/b^2z^2 = 1$, which is an ellipse centered around the z-axis. The axes of the ellipse have lengths $2az/c$ and $2bz/c$.

(b) When $x = 0$, the equation is $y^2/b^2 = z^2/c^2$ or $y = \pm bz/c$. When $y = 0$, the equation is $x^2/a^2 = z^2/c^2$ or $x = \pm az/c$. In both cases, the cross-sections are two straight lines.

25. (c) If the surface contains (x_0,y_0,z_0), then $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c = 0$. The line through $(0,0,0)$ and (x_0,y_0,z_0) is given by $(x,y,z) = t(x_0,y_0,z_0)$. The point (tx_0,ty_0,tz_0) satisfies the original equation as follows: $(tx_0)^2/a^2 + (ty_0)^2/a^2 + (tz_0)^2/c^2 = t^2(x_0^2/a^2 + y_0^2/b^2 + z_0^2/c) = t^2(0) = 0$. Therefore, the entire line through $(0,0,0)$ and (x_0,y_0,z_0) lies on this surface if (x_0,y_0,z_0) is on it.

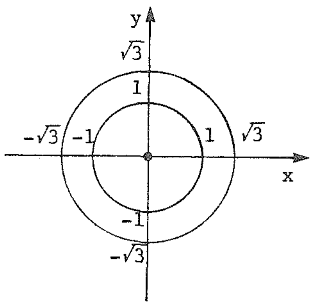
SECTION QUIZ

1. Sketch the following surfaces:
   (a) $x^2/4 + y^2/9 + z^2/16 = 1$
   (b) $9x^2 + 9y^2 - z^2 = 4$
   (c) $9x^2 + 9y^2 - z^2 = -4$
   (d) $3x^2 + 2y^2 - z = 0$
   (e) $4x^2 + y^2/9 = z^2/4$

2. An alien spacecraft has just crash-landed in your neighbor's backyard. These feather-covered aliens wish to have their ship's cracked shell repaired. Unfortunately, your neighbor doesn't understand the cryptic message given to him. The message reads: " σHελλ: $x^2 + y^2 + z^2 = 1$ if $z \leqslant 0$; $4x^2 + 4y^2 + z^2 = 4$ if $z > 0$. " Help your neighbor sketch the shell so that he can help the chicken-like aliens repair their ship.

ANSWERS TO PREREQUISITE QUIZ

1.



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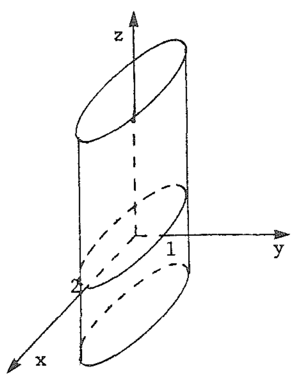
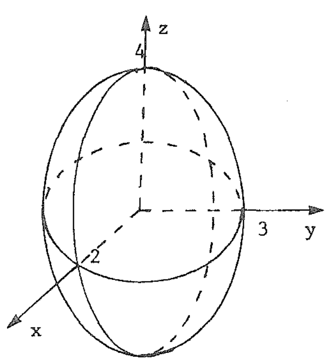


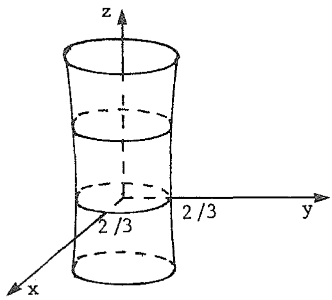
3. One of the variables, x , y , or z , is missing from the equation.

ANSWERS TO SECTION QUIZ

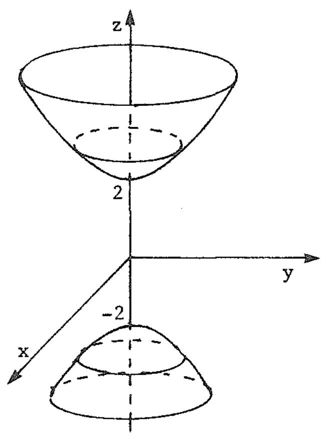
1. (a)



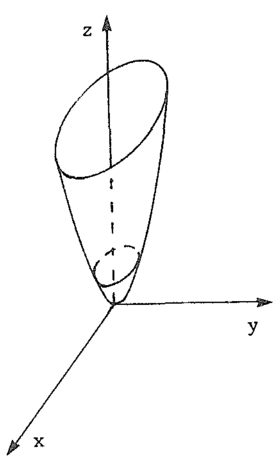
1. (b)



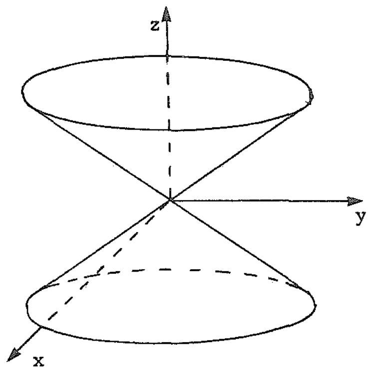
(c)



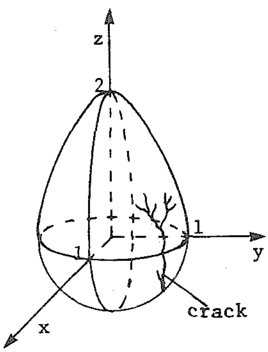
(d)



1. (e)



2.



14.5 Cylindrical and Spherical Coordinates

PREREQUISITES

1. Recall the relationships between cartesian and polar coordinates (Section 5.1).
2. Recall how to graph in polar coordinates (Section 5.6).

PREREQUISITE QUIZ

1. Given a point (x,y), what two equations are used to convert the point to polar coordinates.
2. Convert the polar coordinates $(2,3\pi/4)$ to cartesian coordinates?
3. Sketch the graph of $r = \cos 3\theta$ in the xy-plane.

GOALS

1. Be able to convert from cartesian to cylindrical coordinates, and vice versa.
2. Be able to convert from cartesian to spherical coordinates, and vice versa.

STUDY HINTS

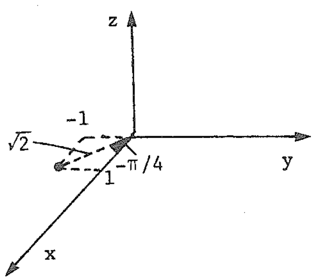
1. Cylindrical coordinates. You should know how to convert back and forth between cylindrical and cartesian coordinates. The formulas to convert from cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. To convert from cartesian coordinates, use $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ if $x \geqslant 0$, $\theta = \tan^{-1}(y/x) + \pi$ if $x < 0$. Plotting the point in xy-coordinates will help you decide what θ should be.

2. Spherical coordinates. Know how to convert back and forth between spherical and cartesian coordinates. In addition, you should know the geometry associated with the variables ρ , ϕ , and θ : ρ is the distance from the origin; ϕ is the angle from the positive z-axis; θ is the same angle as in cylindrical coordinates. The formulas you need to know are $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$, $\rho = \sqrt{x^2 + y^2 + z^2}$, and $\phi = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$. θ is the same as in cylindrical coordinates. Again, plotting the point in xyz-coordinates will help you decide what θ should be.

3. Graphs of r = constant. Note that r = constant in cylindrical coordinates describes a cylinder and that r = constant in spherical coordinates describes a shpere. You may have suspected this from the name of the coordinate system.

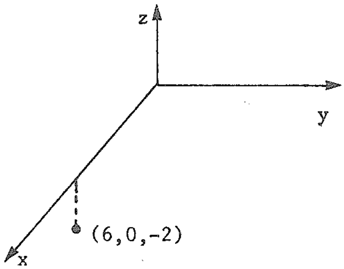
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



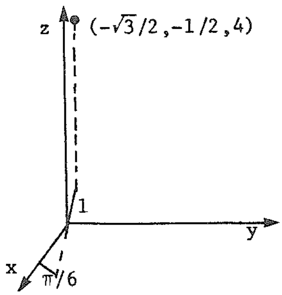
Since $x \geqslant 0$, we use $\theta = \tan^{-1}(y/x)$ rather than $\theta = \tan^{-1}(y/x) + \pi$. Also, $r = \sqrt{x^2 + y^2}$ and $z = z$. Thus, $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $\theta = \tan^{-1}(-1) = -\pi/4$. Therefore, $(1,-1,0)$ converts to $(\sqrt{2},-\pi/4,0)$.

5.



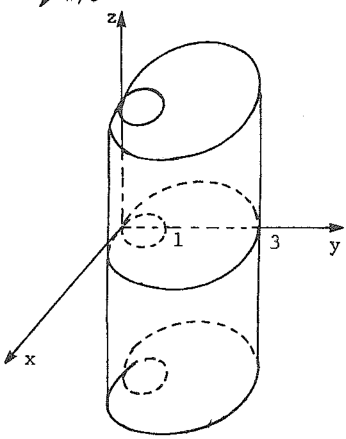
Using the same formulas as in Exercise 1, we get $r = \sqrt{6^2 + 0^2} = 6$ and $\theta = \tan^{-1}(0) = 0$. Therefore, $(6,0,-2)$ converts to $(6,0,-2)$.

9.



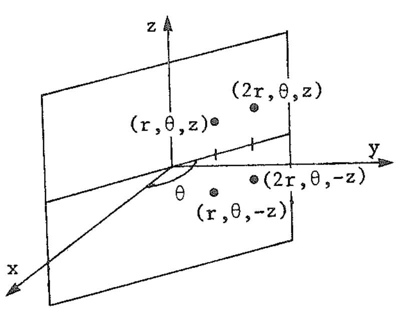
Use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. In this case, $x = (-1)\cos(\pi/6) = -\sqrt{3}/2$ and $y = (-1)\sin(\pi/6) = -1/2$. Thus, $(-1,\pi/6,4)$ converts to $(-\sqrt{3}/2,-1/2,4)$.

13.



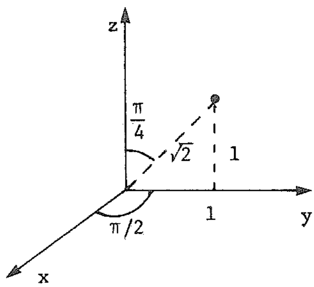
Since z is missing, $r = 1 + 2\cos\theta$ represents a cylinder in the z-axis direction. Note that $r = 1 + 2\cos\theta$ in polar coordinates looks like a cardioid.

17.



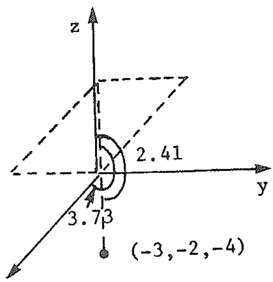
The effect of replacing z by $-z$ is to reflect the point across the xy-plane. The effect of replacing r by $2r$ is to move the point twice as far from the z-axis. Replacing (r,θ,z) by $(2r,\theta,-z)$ moves the points twice as far from the z-axis and reflects it across the xy-plane.

21.



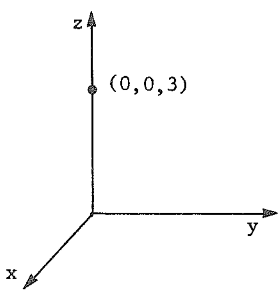
$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 1 + 1} = \sqrt{2}$; $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(1/\sqrt{2}) = \pi/4$. θ is the rotation in the xy-plane from the positive x-axis. In this case, $\theta = \tan^{-1}(1/0) = \tan^{-1}(\pm\infty) = \pi/2$. Thus, $(0,1,1)$ converts to $(\sqrt{2},\pi/2,\pi/4)$.

25.



Since $x < 0$, we use $\theta = \tan^{-1}(y/x) + \pi$, $\rho = \sqrt{x^2 + y^2 + z^2}$, and $\phi = \cos^{-1}(z/\rho)$. Thus, $\rho = \sqrt{9 + 4 + 16} = \sqrt{29}$, $\theta = \tan^{-1}(-2/-3) + \pi \approx 3.73$, and $\phi = \cos^{-1}(-4/\sqrt{29}) \approx 2.41$. Thus, $(-3,-2,-4)$ converts to $(\sqrt{29},3.73,2.41)$.

29.

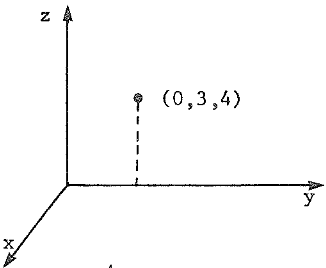


Use $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$. Thus, $x = 3 \sin(0) \cos(2\pi) = 0$, $y = 3 \sin(0) \sin(2\pi) = 0$, and $z = 3 \cos(0) = 3$. Therefore, $(3,\pi/2,0)$ converts to $(0,0,3)$.

33. Substituting $x = \rho \sin\phi \cos\theta$ and $z = \rho \cos\phi$ converts $xz = 1$ to $\rho^2 \sin\phi \cos\phi \cos\theta = 1$ in spherical coordinates. That is, $\rho^2 \sin 2\phi \cos\theta = 2$ or $\rho^2 = 2/\sin 2\phi \cos\theta$. (We used the identity $\sin 2\phi = 2 \sin\phi \cos\phi$.)

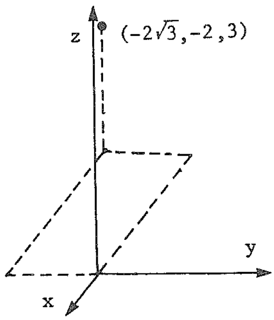
37. Replacing (ρ,θ,ϕ) by $(2\rho,\theta,\phi)$ extends a point twice as far from the origin along the same line through the origin because θ and ϕ do not change.

41. We are given (x,y,z) . The cylindrical coordinates (r,θ,z) are given by $r = \sqrt{x^2 + y^2}$, $\theta = \begin{cases} \tan^{-1}(y/z) & \text{if } x \geqslant 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}$, and $z = z$. The spherical coordinates (ρ,θ,ϕ) are given by $\rho = \sqrt{x^2 + y^2 + z^2}$, $\phi = \cos^{-1}(z/\rho)$ and θ is given by the same formula as the cylindrical coordinate.



$r = \sqrt{0^2 + 3^2} = 3$; $\theta = \tan^{-1}(3/0)$ implies $\theta = \pi/2$; $\rho = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$; $\phi = \cos^{-1}(4/5) \approx 0.64$. Therefore, the cylindrical coordinates are $(3,\pi/2,4)$ and the spherical coordinates are $(5,\pi/2,0.64)$.

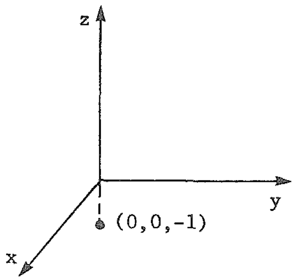
45.



Using the same formulas as in Exercise 41, we get $r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$; $\theta = \tan^{-1}(-2/(-2\sqrt{3})) + \pi = \tan^{-1}(1/\sqrt{3}) + \pi = 7\pi/6$; $\rho = \sqrt{(-2\sqrt{3})^2 + (-2)^2 + 3^2} = \sqrt{25} = 5$; $\phi = \cos^{-1}(3/5) \approx 0.93$. Therefore, the cylindrical coordinates are $(4,7\pi/6,3)$ and the spherical coordinates are $(5,7\pi/6,0.93)$.

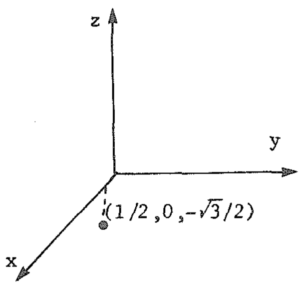
49. Given (r,θ,z) , the conversion formulas for cartesian coordinates are $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The conversion formulas for spherical coordinates are $\rho = \sqrt{r^2 + z^2}$, $\theta = \theta$, and $\phi = \cos^{-1}(z/\rho)$. Therefore, we have $x = (0)(\cos(\pi/4)) = 0$; $y = (0)(\sin(\pi/4)) = 0$; $\rho = \sqrt{0^2 + 1^2} = 1$; $\phi = \cos^{-1}(1/1) = 0$. Thus, the cartesian coordinates are $(0,0,1)$ and the spherical coordinates are $(1,\pi/4,0)$.

53. We are given the spherical coordinates (ρ,θ,ϕ) . The cartesian coordinates are given by $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$. After finding the cartesian coordinates, the cylindrical coordinates are given by $r = \sqrt{x^2 + y^2}$, $\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}$, $z = z$.



$x = (1)\sin(\pi)\cos(\pi/2) = 0$; $y = (1)\sin(\pi)\sin(\pi/2) = 0$; $z = (1)\cos(\pi) = -1$, so the cartesian coordinates are $(0,0,-1)$. $r = \sqrt{0^2 + 0^2} = 0$, and since there is no rotation in the xy-plane, θ can have any value, so the cylindrical coordinates are $(0,\theta,-1)$.

57.

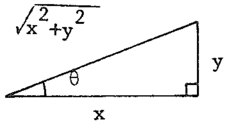


Using the same formulas as in Exercise 53, we get $x = (-1)\sin(\pi/6)\cos(\pi) = (-1)(1/2)(-1) = 1/2$; $y = (-1)\sin(\pi/6) \times \sin(\pi) = 0$; $z = (-1)\cos(\pi/6) = -\sqrt{3}/2$; $r = \sqrt{(1/2)^2 + 0^2} = 1/2$; $\theta = \tan^{-1}(0/(1/2)) = 0$. Therefore, the cartesian coordinates are $(1/2,0,-\sqrt{3}/2)$ and the cylindrical coordinates are $(1/2,0,-\sqrt{3}/2)$.

61. (a) The length of $x\underline{i} + y\underline{j} + z\underline{k}$ is $\sqrt{x^2 + y^2 + z^2}$. The formula for the vector's length is precisely the definition of the spherical coordinate ρ .

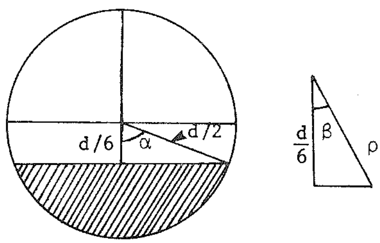
(b) By the definition of the dot product, $\underline{v}\cdot\underline{k} = x(0) + y(0) + z(1) = z$. Also, $\|\underline{v}\| = \sqrt{x^2 + y^2 + z^2}$, so $\cos^{-1}(\underline{v}\cdot\underline{k}/\|\underline{v}\|) = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$, which is the definition of the spherical coordinate ϕ .

61. (c)



$\underline{u}\cdot\underline{i} = x(1) + y(0) + 0(0) = x$, and $\|\underline{u}\| = \sqrt{x^2 + y^2 + 0^2}$, so $\cos^{-1}(\underline{u}\cdot\underline{i}/\|\underline{u}\|) = \cos^{-1}(x/\sqrt{x^2 + y^2})$. From the diagram, we see that $\cos^{-1}(x/\sqrt{x^2 + y^2}) = \theta = \tan^{-1}(y/x)$, which is the definition of θ given in the text.

65.



Note that ϕ will be between $\pi/2$ and π because the region lies in the lower hemisphere. From the triangle, we see that $\cos\alpha = (d/6)/(d/2) = 1/3$; therefore, we have $\pi - \alpha \leqslant \phi \leqslant \pi$ or $\pi - \cos^{-1}(1/3) \leqslant \phi \leqslant \pi$. Now, ρ can be as large as $d/2$; however, as ρ gets smaller, its lower limit depends on ϕ . Pick any ϕ , then $\phi + \beta = \pi$ and according to the diagram $\cos\beta = (d/6)/\rho$. Rearrangement gives $d/6 \cos\beta = \rho = d/6 \cos(\pi - \phi) = -d/6 \cos\phi$. Therefore, $-d/6 \cos\phi \leqslant \rho \leqslant d/2$. So far, we have described the cross-section in one quadrant. The entire volume requires a revolution around the z-axis, so its description is $-d/6 \cos\phi \leqslant \rho \leqslant d/2$, $0 \leqslant \theta \leqslant 2\pi$, and $\pi - \cos^{-1}(1/3) \leqslant \phi \leqslant \pi$.

SECTION QUIZ

1. Explain the geometric meaning of the spherical coordinates ρ , θ , and ϕ
2. The cartesian coordinates $(-1,1,-5/2)$ do not convert to the cylindrical coordinates $(\sqrt{2},-\pi/4,-5/2)$ even though $\tan^{-1}(-1) = -\pi/4$. Why?

3. If a point lies on the xy-plane, then ϕ is: (choose one)

   (a) unknown because we don't know x and y.

   (b) $\pi/2$.

   (c) 90.

4. An inexperienced sailor has become ship-wrecked on a Greek island and has been unable to radio for help. However, upon exploring the island, the sailor comes upon an odd-looking capped bottle. As he uncaps the bottle, he is surprised by an emerging genie. Her last master was Euclid, and she only understands how to locate places with cylindrical coordinates. The sailor knows from his latitude and longitude charts that the spherical coordinates of New York City are $(1,-72^\circ,49^\circ)$. (Distances are measured in Earth-radius units.)

   (a) What are the corresponding cylindrical coordinates?

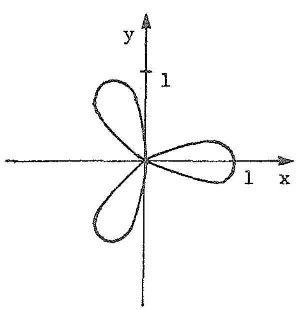
   (b) What are the corresponding cartesian coordinates?

ANSWERS TO PREREQUISITE QUIZ

1. $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$ if $x \geqslant 0$, $\theta = \tan^{-1}(y/x) + \pi$ if $x < 0$.

2. $(-\sqrt{2},\sqrt{2})$

3.



ANSWERS TO SECTION QUIZ

1. ρ is the distance from the origin; θ is the angle in the xy-plane, measured counterclockwise from the x-axis; ϕ is the angle measured from the positive z-axis.

2. θ should be $3\pi/4$.

3. b

4. (a) (0.75,-1.26,0.66)

   (b) (0.23,-0.71,0.66)

14.6 Curves in Space

PREREQUISITES

1. Recall how to sketch curves in the plane described by parametric equations (Sections 2.4 and 10.4).
2. Recall the basic rules for differentiation (Chapters 1 and 2).

PREREQUISITE QUIZ

1. Sketch the curves described by the following sets of parametric equations:

   (a) $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$

   (b) $y = 2t$, $x = 3t - 1$

2. Differentiate the following functions:

   (a) $(x^3 + x)(x^2 + 1)$

   (b) $x/(x - 1)$

   (c) $\sin(3t - 1)$

GOALS

1. Be able to sketch curves in space.
2. Be able to differentiate vector functions.

STUDY HINTS

1. Parametric lines. One of the simplest curves to write parametrically is the straight line. As shown by Example 2, the parameter occurs with only one power. In (a), t occurs only to the first power. In (b), t occurs in the third power.
2. Sketching curves. As with surfaces, we sketch curves with a technique similar to the method of sections. In many cases, the curve is drawn in the xy-plane and then lifted to the appropriate height.

3. Terminology. The object $(f(t),g(t),h(t))$ may be called a parametric curve, a vector function, or a triple of scalars.

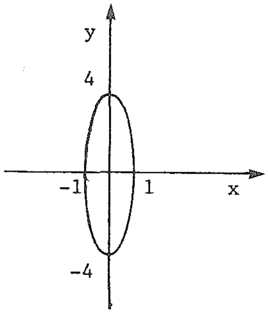
4. Differentiating vector functions. The derivative of $\underline{\sigma}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ is $\underline{\sigma}'(t) = f'(t)\underline{i} + g'(t)\underline{j} + h'(t)\underline{k}$. This formula should be memorized.

5. Differentiation rules. In the box on p. 740, note that there are three product rules for vectors — scalar multiplication, dot product, and cross product. These rules are very similar to the rules you learned in Chapters 1 and 2.

6. Physical application. Suppose that $(f(t),g(t),h(t))$ describes a parametric curve. The derivative $(f'(t),g'(t),h'(t))$ is known as a velocity vector and the second derivative is the acceleration vector. The length of the velocity vector is called the speed. This is analogous to the material presented in Section 10.4.

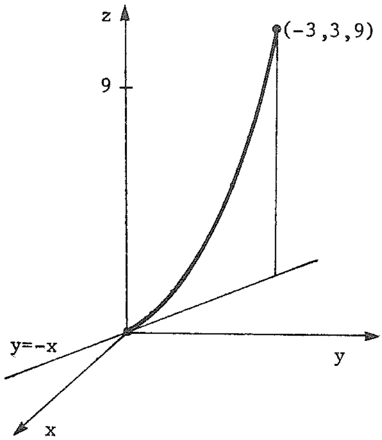
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



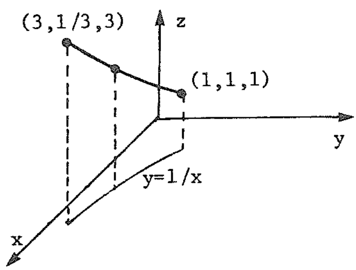
Rearrangement gives $y/4 = \cos t$; therefore, $x^2 + (y/4)^2 = \sin^2 t + \cos^2 t = 1$. This is an ellipse with intercepts $(\pm 1,0)$ and $(0,\pm 4)$. The curve is traced out clockwise starting at $(0,4)$.

5.



Note that $x = -t$, $y = t$ represents the straight line $y = -x$ in the xy-plane. Thus, $(x,y,z) = (-t,t,t^2)$ moves along a curve over this line with the z component moving like t^2.

9.



In the xy-plane, we have the hyperbola $xy = 1$. Move its points t units in the z-direction to get the desired curve.

13. If $\underline{\sigma}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$, then $\underline{\sigma}'(t) = f'(t)\underline{i} + g'(t)\underline{j} + h'(t)\underline{k}$. Since $\underline{\sigma}(t) = 3(\cos t)\underline{i} - 8(\sin t)\underline{j} + e^t\underline{k}$, we get $\underline{\sigma}'(t) = -3(\sin t)\underline{i} - 8(\cos t)\underline{j} + e^t\underline{k}$. Similarly, $\underline{\sigma}''(t) = -3(\cos t)\underline{i} + 8(\sin t)\underline{j} + e^t\underline{k}$.

17. By the cross-product rules for differentiation, $(d/dt)[\underline{\sigma}_1(t) \times \underline{\sigma}_2(t)] = \underline{\sigma}_1'(t) \times \underline{\sigma}_2(t) + \underline{\sigma}_1(t) \times \underline{\sigma}_2'(t) = [t^3\cos t(-2 + \csc^2 t) - 3t^2\sin t\ (2 + \csc^2 t)]\underline{i} + [t^2e^{-t}(3 - t) + 2t^2e^t(t + 3)]\underline{j} + [e^t \csc t(1 - \cot t) - e^{-t}(\cos t + \sin t)]\underline{k}$. On the other hand, $\underline{\sigma}_1(t) \times \underline{\sigma}_2(t) = (-2t^3\sin t - t^3\csc t)\underline{i} + (t^3e^{-t} + 2t^3e^t)\underline{j} + (e^t\csc t - e^{-t}\sin t)\underline{k}$. Thus, $(d/dt)[\underline{\sigma}_1(t) \times \underline{\sigma}_2(t)] = [(d/dt)(-2t^3\sin t - t^3\csc t)]\underline{i} + [(d/dt)(t^3e^{-t} + 2t^3e^t)]\underline{j} + [(d/dt)(e^t\csc t - e^{-t}\sin t)]\underline{k}$. Completing the differentiation yields the same result as before.

21. Let $\underline{\sigma}(t)$ be the path of an object. Then $\underline{\sigma}'(t)$ is the velocity and $\underline{\sigma}''(t)$ is the acceleration vector of the object. We are given that $\underline{\sigma}'(t)\cdot\underline{\sigma}''(t) = 0$ for all t. Since $(1/2)(d/dt)\|\underline{\sigma}'(t)\|^2 = \underline{\sigma}'(t)\cdot\underline{\sigma}''(t) = 0$, we have $(\underline{\sigma}'\cdot\underline{\sigma}')(t) = \|\underline{\sigma}'(t)\|^2$ being a constant, i.e., the speed of the object is constant.

25. The velocity vector of $f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ is $\underline{v} = f'(t)\underline{i} + g'(t)\underline{j} + h'(t)\underline{k}$. The acceleration vector is $\underline{a} = f''(t)\underline{i} + g''(t)\underline{j} + h''(t)\underline{k}$. The speed is $[(f'(t))^2 + (g'(t))^2 + (h'(t))^2]^{1/2}$. In this case, $f(t) = 2t - 1$, $g(t) = t + 2$, and $h(t) = t$. Thus, $\underline{v} = 2\underline{i} + \underline{j} + \underline{k}$, $\underline{a} = \underline{0}$, and the speed is $\sqrt{4 + 1 + 1} = \sqrt{6}$.

29. Using the same formulas as in Exercise 25 with $f(t) = 4\cos t$, $g(t) = 2\sin t$, and $h(t) = t$, we get $\underline{v} = -4(\sin t)\underline{i} + 2(\cos t)\underline{j} + \underline{k}$, $\underline{a} = -4(\cos t)\underline{i} - 2(\sin t)\underline{j}$, and the speed is $\sqrt{16\sin^2 t + 4\cos^2 t + 1} = \sqrt{5 + 12\sin^2 t}$.

33. For a parametric curve $(f(t),g(t),h(t))$, the velocity vector is $(f'(t),g'(t),h'(t))$ and the acceleration vector is $(f''(t),g''(t),h''(t))$. The tangent line is given by $(f(t_0),g(t_0),h(t_0)) + (t - t_0)(f'(t_0), g'(t_0),h'(t_0))$. In this case, $\underline{v}(t) = (6,6t,3t^2)$; $\underline{a}(t) = (0,6,6t)$; therefore, the tangent line is $(0,0,0) + (6,0,0)t = (6,0,0)t$.

37. Using the same method as in Exercise 33, we get $\underline{v}(t) = (\sqrt{2},e^t,-e^{-t})$; $\underline{a}(t) = (0,e^t,e^{-t})$; therefore, the tangent line is $(0,1,1) + (\sqrt{2},1,-1)t$.

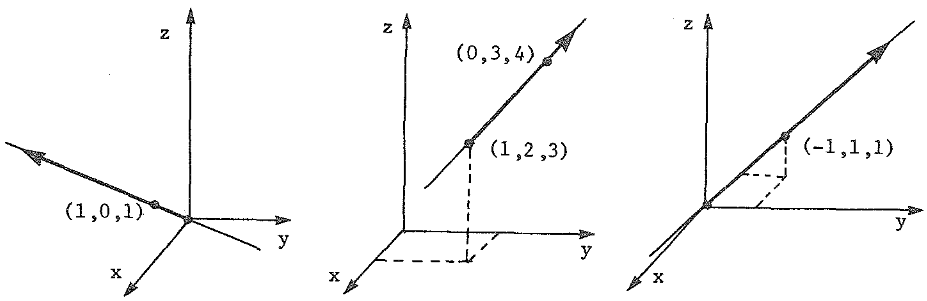
41. We integrate $\underline{\sigma}'(t) = (A,B,C)$ to get $\underline{\sigma}(t) = (At + c_1,Bt + c_2,Ct + c_3)$. Substituting $t = 0$ gives $\underline{\sigma}(0) = (c_1,c_2,c_3)$. Thus, we get the following curves:

41. (continued)

(a) $\underline{\sigma}(t) = (1,0,1)t$.

(b) $\underline{\sigma}(t) = (1,2,3) + (-1,1,1)t$.

(c) $\underline{\sigma}(t) = (-1,1,1)t$.

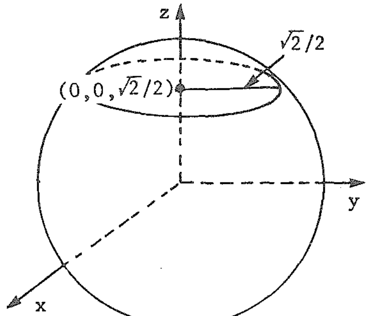


(a) (b) (c)

45. (a) Since $\underline{\sigma}_1(t)$ and $\underline{\sigma}_2(t)$ satisfies the differential equation, then it must be true that $\underline{\sigma}_1''(t) = -\underline{\sigma}_1(t)$ and $\underline{\sigma}_2''(t) = -\underline{\sigma}_2(t)$. Now, let $\underline{\sigma}(t) = A_1\underline{\sigma}_1(t) + A_2\underline{\sigma}_2(t)$. We have $\underline{\sigma}''(t) = A_1\underline{\sigma}_1''(t) + A_2\underline{\sigma}_2''(t) = -A_1\underline{\sigma}_1(t) - A_2\underline{\sigma}_2(t) = -\underline{\sigma}(t)$.

(b) Section 8.1 provides the solution to the spring equation $d^2x/dt^2 = -\omega^2 x$ as $x = A\cos\omega t + B\sin\omega t$, for constants A and B . Here, $\omega = 1$, and each component of $\underline{\sigma}(t)$ must satisfy the spring equation, so the most general form of $\underline{\sigma}(t)$ is $(A_1\cos t + B_1\sin t, A_2\cos t + B_2\sin t, A_3\cos t + B_3\sin t)$. A_1 , A_2 , A_3 , B_1 , B_2 , and B_3 are all constants.

49. (a)

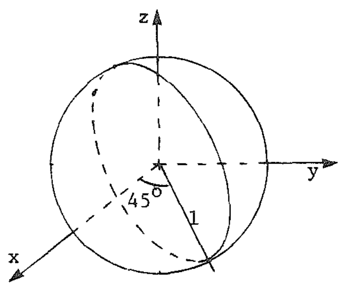


$(x^2 + y^2) + z^2 = \sin^2\phi(\cos^2 t + \sin^2 t) + \cos^2\phi = \sin^2\phi + \cos^2\phi = 1$, so the curve lies on the unit sphere. Since ϕ is fixed, the curve lies parallel to the xy-plane; therefore, the curve is a circle. The center is at $(0,0,\cos\phi)$

49. (a) (continued)

and the radius is $\sin\phi$. For $\phi = 45^\circ$, the circle lies in the plane $z = \sqrt{2}/2$.

(b)



$(x^2 + y^2) + z^2 = \sin^2 t(\cos^2\theta + \sin^2\theta) + \cos^2 t = \sin^2 t + \cos^2 t = 1$, so the curve lies on the unit sphere. Since θ is fixed, we have $x/y = \cos\theta/\sin\theta$ or $y = x\tan\theta$, which is a straight line segment of length 2 with slope $\tan\theta$ in the xy-plane. As t varies with $\theta = 45^\circ$, the curve moves in a circle over the line $x = y$ beginning from $(0,0,1)$.

53. Let $\underline{\sigma}_1(t) = f_1(t)\underline{i} + g_1(t)\underline{j} + h_1(t)\underline{k}$ and $\underline{\sigma}_2(t) = f_2(t)\underline{i} + g_2(t)\underline{j} + h_2(t)\underline{k}$. Then $\underline{\sigma}_1(t) + \underline{\sigma}_2(t) = [f_1(t) + f_2(t)]\underline{i} + [g_1(t) + g_2(t)]\underline{j} + [h_1(t) + h_2(t)]\underline{k}$, and $(d/dt)[\underline{\sigma}_1(t) + \underline{\sigma}_2(t)] = (d/dt)[f_1(t) + f_2(t)]\underline{i} + (d/dt)[g_1(t) + g_2(t)]\underline{j} + (d/dt)[h_1(t) + h_2(t)]\underline{k} = [f_1'(t) + f_2'(t)]\underline{i} + [g_1'(t) + g_2'(t)]\underline{j} + [h_1'(t) + h_2'(t)]\underline{k}$ by the sum rule for real-valued functions. Rearranging the terms, we get $(d/dt)[\underline{\sigma}_1(t) + \underline{\sigma}_2(t)] = \underline{\sigma}_1'(t) + \underline{\sigma}_2'(t)$.

57. (a) $\underline{\sigma}(t)\cdot\underline{u} = 0$ implies $\|\underline{\sigma}(t)\|\|\underline{u}\|\cos\theta = 0$ so $\theta = \pm\pi/2$ (provided $\underline{\sigma}(t) \neq \underline{0}$). Therefore, $\underline{\sigma}(t)$ is always perpendicular to $\underline{u}$, and $\underline{\sigma}(t)$ describes any curve in a plane through the origin and perpendicular to $\underline{u}$.

(b) By differentiating, we get $\underline{\sigma}'(t)\cdot\underline{u} + \underline{\sigma}(t)\cdot\underline{u}' = 0$. Since $\underline{u}$ is constant, $\underline{u}' = 0$, and $\underline{\sigma}'(t)\cdot\underline{u} = 0$. Thus, $\underline{\sigma}'(t)$ is always perpendicular to $\underline{u}$. Again, this implies that $\underline{\sigma}(t)$ is any curve in a plane perpendicular to $\underline{u}$. This plane need not go through the origin.

57. (c) $\underline{\sigma}(t)\cdot\underline{u} = \|\underline{\sigma}(t)\|\cdot\|\underline{u}\| \cos\theta = b\|\underline{\sigma}(t)\|$. Since $\underline{u}$ is a unit vector the equation reduces to $\cos\theta = b$ or $\theta = \cos^{-1}b$, where θ is between $-\pi/2$ and $\pi/2$. Therefore, $\underline{\sigma}(t)$ is any curve lying in the cone with $\underline{u}$ as its axis and vertex angle $2\cos^{-1}b$.

SECTION QUIZ

1. Sketch the curve $\underline{\sigma}(t) = (t \sin t, t \sin t, t)$.

2. Find $\underline{\sigma}'(t)$ and $\underline{\sigma}''(t)$ for the curve in Question 1.

3. Sketch the curve $\underline{\sigma}(t) = (\sin 2t + 3, \cos 2t + 1, t)$. [Hint: How does this compare to the curve $(\sin 2t, \cos 2t, t)$?]

4. Let $\underline{u}(t) = 3t\underline{i} + e^t\underline{j} + (\cos t)\underline{k}$ and $\underline{v}(t) = (\ln t)\underline{i} - (\sin^{-1}t)\underline{j} + (t/(1-t))\underline{k}$. Compute:

   (a) $(d/dt)(\underline{u}\cdot\underline{v})$

   (b) $(d/dt)(\underline{u}\times\underline{v})$

   (c) $(d/dt)\underline{u}(t^2)$

   (d) $(d/dy)[\underline{u}(y) - v(y)]$

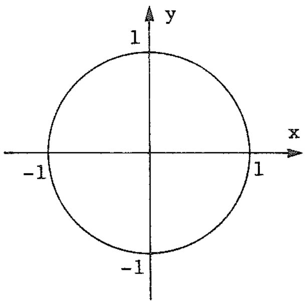
5. Cute Karen, the fairest lady of the land, had been imprisoned in a tall tower by a mean magician. Along comes Prince Charming on his white stallion to the rescue. A sign at the bottom of the stairwell reads: "DETOUR: THIS STAIRCASE $[x = (t-4)\cos t$, $y = (t-4)\sin t$, $z = t^2$, $0 \leqslant t \leqslant 4]$ UNDER CONSTRUCTION. USE ALTERNATE STAIRCASE — $(x = 0$, $y = -4\cos t$, $z = -4|\sin 2t|)$."

   (a) Sketch the path of the old route.

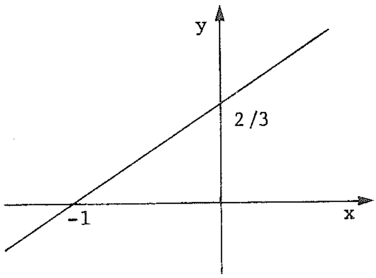
   (b) At $t = \pi$, another sign reads: "EXIT TO HOSTAGE ROOM. USE $x = 0$, $y = 4 - 4t$, $z = 16t$, $0 \leqslant t \leqslant 1$." Finally, when Prince Charming saves Cute Karen, another sign magically appears. It says: "HA! HA! THE PATH HAS COLLAPSED BEHIND YOU. YOU MAY GET TO THE GROUND BY REFLECTING YOUR ROUTE ACROSS THE PLANE $y = 0$. SIGNED, MEAN MAGICIAN." Sketch the entire rescue route.

ANSWERS TO PREREQUISITE QUIZ

1. (a)



(b)



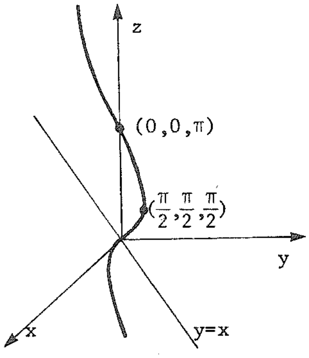
2. (a) $(3x^2 + 1)(x^2 + 1) + (x^3 + x)(2x) = 5x^4 + 6x^2 + 1$

(b) $-1/(x - 1)^2$

(c) $3 \cos(3t - 1)$

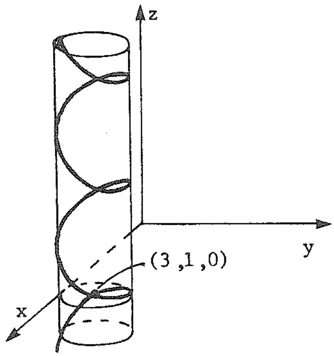
ANSWERS TO SECTION QUIZ

1.



2. $\underline{\sigma}'(t) = (\sin t + t\cos t, \sin t + t\cos t, 1)$; $\underline{\sigma}''(t) = (2\cos t - t\sin t,$
$2\cos t - t\sin t, 0)$

3.



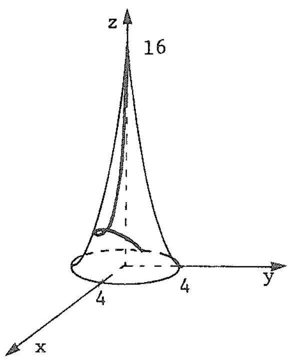
4. (a) $3\ln t + 3 + (1/\sqrt{1-t^2} - \sin^{-1}t)e^t - t\sin t/(1-t) + \cos t/(1-t)^2$

(b) $[te^t/(1-t) - (\sin^{-1}t)(\sin t) - \cos t/\sqrt{1-t^2} + e^t/(1-t)^2]\underline{i} +$
$[(\ln t)(\sin t) - 3t/(1-t) - 3t/(1-t)^2 + \cos t/t]\underline{j} + [-3\sin^{-1}t -$
$(e^t)(\ln t) - e^t/t + 3t/\sqrt{1-t^2}]\underline{k}$

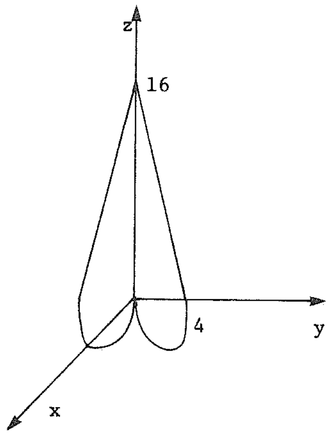
(c) $6t\underline{i} + 2t\exp(t^2)\underline{j} - 2t\sin(t^2)\underline{k}$

(d) $(3 - 1/y)\underline{i} + (e^y - 1/\sqrt{1-y^2})\underline{j} + (1/(1-t)^2 - \sin t)\underline{k}$

5. (a)



5. (b)



14.7 The Geometry and Physics of Space Curves

PREREQUISITES

1. Recall how to compute the length of a curve in the plane described by parametric equations (Section 10.4).
2. Recall Newton's second law (Section 8.1).

PREREQUISITE QUIZ

1. If a curve is described parametrically by $x = t^3 + 2t$ and $y = t^2 + 3$, what is the length of the curve for $0 \leqslant t \leqslant 2$? Express your answer as a definite integral.
2. State Newton's second law.

GOALS

1. Be able to compute the arc length of a vector function.
2. Be able to compute the curvature of a curve.
3. Be able to solve problems involving Newton's second law for curves.

STUDY HINTS

1. Arc length. In Section 10.4, you were given a formula for the arc length of a parametric curve in a plane. For the length of a curve in space, we simply add $(h'(t))^2$ under the radical sign, as you might have expected. Another formula for arc length is $L = \int_a^b \|\underline{\sigma}'(t)\|\, dt$. Thinking of $\|\underline{\sigma}'(t)\|$ as speed, the arc length equation becomes: distance $= \int_a^b(\text{speed})dt$.
2. Arc length integration. A very useful formula is $\int\sqrt{x^2 + a^2}\, dx =$ $(1/2)[x\sqrt{x^2 + a^2} + a^2 \ln(x + \sqrt{x^2 + a^2})] + C$. Keep this in mind for reference, but don't try to memorize it. (Many instructors will state it on an exam or give you a short table of integrals. Ask.) See how it is used in Example 2.

3. Planetary motion. The main point that you should get out of the discussion about Newton's and Kepler's findings is that $T^2 = r_0^3(2\pi)^2/GM$. This is not to be memorized. It just shows how calculus can be applied to the real world.

4. Curvature. Curvature is a rate which tells you how fast the direction of motion is changing. A large curvature is associated with a sharp bend. A straight line has no curvature. And a small circle has large curvature. Since curvature is a rate, it is a derivative.

5. Definitions. Let $\underline{v}(t)$ be the velocity vector of a curve. If $\underline{v} \neq \underline{0}$ for all t, the curve is called regular. $\underline{T} = \underline{v}/\|\underline{v}\|$ is the unit tangent vector, and if $\|\underline{v}\| = 1$ for all t, then the curve is said to be parametrized by arc length. Example 8 shows how regular curves can be converted to curves parametrized by arc length.

6. Curvature formula. This formula only applies to curves parametrized by arc length. It is $k = \|d\underline{T}/ds\|$, where s is the parameter we use rather than t when a curve is parametrized by arc length.

7. More definitions. The principal normal vector is $\underline{N} = (d\underline{T}/ds)/k = (d\underline{T}/ds)/\|d\underline{T}/ds\|$. It is orthogonal to $\underline{T}$ and points from the concave side of the curve. See Fig. 14.7.4.

8. Supplement (Optional). The purpose of the supplement is to derive the sunshine formula (number 6 on p. 759) using vectors and rotations. Formula (6) and its derivation are not intended to be memorized. However, if you enjoy mathematics, you may like this derivation. Treat it as fun!

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The arc length L is $\int_a^b[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]^{1/2}dt$. In this case, the length is $\int_0^{2\pi}[(-2\sin t)^2 + (2\cos t)^2 + 1]^{1/2}dt = \int_0^{2\pi}\sqrt{5}\,dt = 2\pi\sqrt{5}$.

5. The length is $\int_1^2\sqrt{1 + 1 + (2t)^2}\,dt = \int_1^2\sqrt{2 + 4t^2}\,dt = 2\int_1^2\sqrt{1/2 + t^2}\,dt$. This has the form $\int\sqrt{x^2 + a^2}$, where $a = \sqrt{1/2}$. Thus, $L = 2(1/2) \times [t\sqrt{t^2 + 1/2} + (1/2)\ln(t + \sqrt{t^2 + 1/2})]\Big|_1^2 = 2\sqrt{9/2} - \sqrt{3/2} + (1/2) \times [\ln(2 + \sqrt{9/2}) - \ln(1 + \sqrt{3/2})] = (6\sqrt{2} - \sqrt{6})/2 + (1/2)\ln[(2\sqrt{2} + 3)/(\sqrt{2} + \sqrt{3})] \approx 3.326$.

9. The period is given by $T = \sqrt{r_0^3(2\pi)^2/GM}$. Let R be the radius of the earth in meters. 500 miles $\approx 8.05 \times 10^5$ meters, so $T = \sqrt{(R + 8.05 \times 10^5)^3(2\pi)^2/GM}$. Using the constants from Example 5, we get $T = [(7.17 \times 10^6)^3(2\pi)^2/(6.67 \times 10^{-11})(5.98 \times 10^{24})]^{1/2} \approx (6.50 \times 10^3)$ seconds.

13. (a) $\underline{F} = m\underline{a} = m\underline{\sigma}''(t) = m(x''\underline{i} + y''\underline{j} + z''\underline{k}) = (q/c)\underline{v} \times \underline{B} = [(q/c)(x'\underline{i} + y'\underline{j} + z'\underline{k})] \times b\underline{k} = (qb/c)y'\underline{i} - (qb/c)x'\underline{j}$. Equating the $\underline{i}$, $\underline{j}$, and $\underline{k}$ components, we get $x'' = (qb/cm)y'$; $y'' = -(qb/cm)x'$; $z'' = 0$.

(b) Integrating $x'' = (qb/mc)y'$ yields $x' + C = (qb/mc)y$. When $t = 0$, $x' = 0$ and $y = 0$ by statement (2). Thus, $C = 0$. Substituting into the equation for y'' , we obtain $y'' = -(qb/mc)^2y$, which is the spring equation with $\omega = qb/mc$. We have $y_0 = 0$ and $y_0' = a$, so $y = y_0\cos\omega t + (y_0'/\omega)\sin\omega t = (amc/qb)\sin(qbt/mc)$.

Substituting for y , we get $x' = a\sin(qbt/mc)$, and integration yields $x + C = -(amc/qb)\cos(qbt/mc)$. When $t = 0$, $x = 1$, so $C = -amc/qb - 1$; therefore, $x = -(amc/qb)\cos(qbt/mc) +$

13\. (b) (continued)

$amc/qb + 1$.

Integration of $z'' = 0$ yields $z' = C$, but $z'_0 = c$ implies $z' = c$. Integrating again gives $z = ct + C$. Since $z_0 = 0$, $z = ct$.

(c) In the xy-plane, we have $(x - amc/qb - 1)^2 + y^2 = (-amc/qb)^2\cos^2(qbt/mc) + (amc/qb)^2\sin^2(qbt/mc) = (amc/qb)^2$. This is a circle of radius amc/qb centered at $(amc/qb + 1,0)$. The z-component describes the up and down motion, so the path is a right circular helix. The axis of the cylinder is parallel to the z-axis and passes through the point $(amc/qb + 1,0,0)$.

17\. The ellipse $x^2 + 2y^2 = 1$, $z = 0$ can be parametrized by $\underline{\sigma}(t) = (\cos t, \sin t/\sqrt{2}, 0)$. Then $\underline{v} = \underline{\sigma}'(t) = (-\sin t,(1/\sqrt{2})\cos t,0)$ and $\underline{v}' = (-\cos t,-(1/\sqrt{2})\sin t,0)$. Thus, the curvature is $k = \|\underline{v} \times \underline{v}'\|/\|\underline{v}\|^3 = \|(0,0,(1/\sqrt{2})\sin^2 t + (1/\sqrt{2})\cos^2 t)\|/\|(-\sin t,(1/\sqrt{2})\cos t,0)\|^3 = 1/\sqrt{2}(\sin^2 t + \cos^2 t/2)^{3/2} = 1/\sqrt{2}(2y^2 + x^2/2)^{3/2}$.

21\. The particle is moving at constant speed, so let $\|\underline{v}\| = c$. Reparametrize by arc length using $s = ct$. Then $\underline{\sigma}(t) = \underline{\sigma}(s/c) = p(s)$, and so the unit tangent vector is $\underline{T} = p'(s) = (1/c)\underline{\sigma}'(s/c)$, where $\underline{\sigma}'(s/c)$ is the velocity. Then $d\underline{T}/ds = (1/c^2)\underline{\sigma}''(s/c)$, where $\underline{\sigma}''(s/c)$ is the acceleration. Now, by definition, $k = \|d\underline{T}/ds\|$ and $\underline{N} = (d\underline{T}/ds)/\|d\underline{T}/ds\|$. Therefore, the force is $\underline{F} = m\underline{a} = mc^2(d\underline{T}/ds)$, and so, $\|\underline{F}\| = mc^2 k$.

25\. By definition, $\underline{N} = (d\underline{T}/ds)/\|d\underline{T}/ds\| = (d\underline{T}/ds)/k$. Rearrangement gives the first equation: $d\underline{T}/ds = k\underline{N}$. Next, using the chain rule, we get $d\underline{B}/ds = (d\underline{B}/dt)\cdot(dt/ds) = (-\tau\|\underline{v}\|\underline{N})\cdot(dt/ds)$. But $dt/ds = 1/(ds/dt) = 1/\|\underline{v}\|$, so $d\underline{B}/ds = (-\tau\|\underline{v}\|\underline{N})\cdot\|\underline{v}\| = -\tau\underline{N}$, which is the third equation.

25. (continued).

$d\underline{N}/ds$ has the form $x\underline{T} + y\underline{N} + z\underline{B}$. Take the dot product with $\underline{T}$ to get $(d\underline{N}/ds)\cdot\underline{T} = x(\underline{T}\cdot\underline{T}) + y(\underline{N}\cdot\underline{T}) + z(\underline{B}\cdot\underline{T})$. Since $\underline{B}$, $\underline{T}$, and $\underline{N}$ are mutually orthogonal unit vectors, we get $(d\underline{N}/ds)\cdot\underline{T} = x$. Similarly, $(d\underline{N}/ds)\cdot\underline{B} = x(\underline{T}\cdot\underline{B}) + y(\underline{N}\cdot\underline{B}) + z(\underline{B}\cdot\underline{B}) = z$, and $(d\underline{N}/ds)\cdot\underline{N} = x(\underline{T}\cdot\underline{N}) + y(\underline{N}\cdot\underline{N}) + z(\underline{B}\cdot\underline{N}) = y$. We also know that $0 = d(\underline{N}\cdot\underline{N})/ds = (d\underline{N}/ds)\cdot\underline{N} + \underline{N}\cdot(d\underline{N}/ds) = 2\underline{N}\cdot(d\underline{N}/ds)$ or $(d\underline{N}/ds)\cdot\underline{N} = 0$, so $y = 0$. Also, $0 = d(\underline{N}\cdot\underline{B})/ds = (d\underline{N}/ds)\cdot\underline{B} + \underline{N}\cdot(d\underline{B}/ds) = (d\underline{N}/ds)\cdot\underline{B} + \underline{N}\cdot(-\tau\underline{N}) = (d\underline{N}/ds)\cdot\underline{B} - \tau\|\underline{N}\|^2$. Thus, $(d\underline{N}/ds)\cdot\underline{B} = \tau$, so $z = \tau$. And $0 = d(\underline{N}\cdot\underline{T})/ds = (d\underline{N}/ds)\cdot\underline{T} + \underline{N}\cdot(d\underline{T}/ds) = (d\underline{N}/ds)\cdot\underline{T} + \underline{N}\cdot(k\underline{N}) = (d\underline{N}/ds) + k\|\underline{N}\|^2$, so $x = -k$. Therefore, $d\underline{N}/ds = -k\underline{T} + \tau\underline{B}$.

SECTION QUIZ

1. Find the arc length of the following vector functions:
   (a) $(3t^4 + 3, 8\sqrt{2}t^3, 24t^2 - 13)$, $-1 \leqslant t \leqslant 1$
   (b) $(-2,5,1)$, $2 \leqslant t \leqslant 4$
   (c) $6t^2\underline{i} + 9\underline{j} - 5t\underline{k}$ for $0 \leqslant t \leqslant 3$
2. (a) Suppose the curvature of a vector function is zero, what can you say about the curve?
   (b) Suppose the curvature of a vector function was found to be negative, what can you say about the curve?
3. A ghostly planet has been discovered revolving around the sun at a distance which is twice the Earth-sun distance. An invisibility shield had kept the planet a secret until radar discovered a mass equal to the Earth's at the specified location. How many Earth years does the ghost planet need to orbit the sun?

4. Reckless Roger, the famous car racer, suffers from curviphobia (the fear of curves). He refuses to race on any track which has curvature greater than 2 . Suppose a track is laid down along the curve parametrized by $(x,y,z) = (\cos t, 2\sin t, 0)$.

   (a) Sketch the race track.

   (b) Using the sketch in (a) and your knowledge of curvature, find the points where the curvature is smallest.

   (c) Find the maximum curvature to determine if Reckless Roger will suffer curviphobia on this race track.

ANSWERS TO PREREQUISITE QUIZ

1. $\int_0^2 (9t^4 + 16t^2 + 4)^{1/2} dt$

2. $\underline{F} = m(d^2\underline{x}/dt^2)$

ANSWERS TO SECTION QUIZ

1. (a) 54

   (b) 0

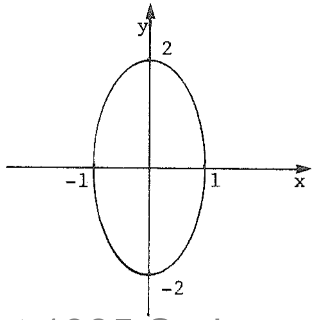
   (c) $3\sqrt{1321}/2 + (25/12) \ln |(36 + \sqrt{1321})/5|$

2. (a) It is a straight line.

   (b) Nothing; an error was made. Curvature is never negative.

3. $2\sqrt{2}$ years

4. (a)



4. (b) $(\pm 1, 0)$

(c) Maximum curvature at $(0, \pm 2) = 2$, so the track is barely acceptable.

14.S Supplement to Chapter 14: Rotations and the Sunshine Formula

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) The angle between $\underline{l}$ and $\underline{r}_0$ is given by $\cos\lambda = \underline{l}\cdot\underline{r}_0 = -1/2$. $\lambda = 2\pi/3$ and $\sin\lambda = \sqrt{3}/2$. From formula (1), we get $\underline{m}_0 = \underline{r}_0/\sin\lambda - \underline{l}\cos\lambda/\sin\lambda = (2/\sqrt{3})(1/\sqrt{2})(\underline{j} + \underline{k}) - (1/\sqrt{3})(1/\sqrt{2})(\underline{i} - \underline{j}) = (1/\sqrt{6})(\underline{i} + \underline{j} + 2\underline{k})$, and $\underline{n}_0 = \underline{l} \times \underline{m}_0 = (1/2\sqrt{3})(\underline{i} + \underline{j} - \underline{k})$.

   (b) From formula (2), $\underline{r} = \underline{\sigma}(t) = \cos\lambda\underline{l} + \sin\lambda\cos(2\pi t/T)\underline{m}_0 + \sin\lambda\sin(2\pi t/T)\underline{n}_0 = (-1/2)(\underline{j} + \underline{k})/\sqrt{2} + (\sqrt{3}/2)\cos(\pi t/12)(1/\sqrt{6}) \times (\underline{i} + \underline{j} + 2\underline{k}) + (\sqrt{3}/2)\sin(\pi t/12)(1/2\sqrt{3})(\underline{i} + \underline{j} - \underline{k}) = [\cos(\pi t/12)/2\sqrt{2} + \sin(\pi t/12)/4]\underline{i} + [-1/2\sqrt{2} + \cos(\pi t/12)/2\sqrt{2} + \sin(\pi t/12)/4]\underline{j} + [-1/2\sqrt{2} + \cos(\pi t/12)/\sqrt{2} - \sin(\pi t/12)/4]\underline{k}$.

   (c) When $t = 12$ and $T = 24$, part (b) tells us that $\underline{\sigma}(12) = (-1/2\sqrt{2})\underline{i} - (1/\sqrt{2})\underline{j} - (3/2\sqrt{2})\underline{k}$. $\underline{\sigma}'(t) = [-(\pi/12)\sin(\pi t/12)/2\sqrt{2} + (\pi/12)\cos(\pi t/12)/4]\underline{i} + [-(\pi/12)\sin(\pi t/12)/2\sqrt{2} + (\pi/12)\cos(\pi t/12)/4]\underline{j} + [-(\pi/12)\sin(\pi t/12)/\sqrt{2} - (\pi/12)\cos(\pi t/12)/4]\underline{k}$, so $\underline{\sigma}'(12) = (-\pi/48)\underline{i} - (\pi/48)\underline{j} + (\pi/48)\underline{k}$. Therefore, the tangent line is $(-1/2\sqrt{2})(\underline{i} + 2\underline{j} + 3\underline{k}) + (-\pi/48)(\underline{i} + \underline{j} + \underline{k})(t - 12)$.

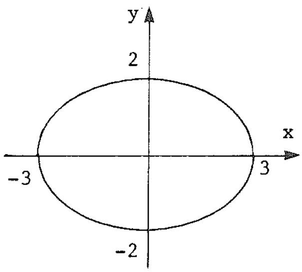
5. Sunset occurs when $\sin A = 0$. Let $B = 2\pi t/T_y$ and $C = 2\pi t/T_d$. Thus, we have $-\cos B(\sin\ell\sin\alpha + \cos\ell\cos\alpha\cos C) = \sin B\cos\ell\sin C$. Dividing by $-\cos B\cos\ell$ gives $\tan\ell\sin\alpha + \cos\alpha\cos C = -\tan B\sin C$, i.e., $\tan\ell\sin\alpha = -\tan B\sin C - \cos\alpha\cos C = -\cos C(\tan B\tan C - \cos\alpha)$. Thus, our "exact" formula is $-\tan\ell\sin\alpha = \cos(2\pi t/T_d)[\tan(2\pi t/T_y)\tan(2\pi t/T_d) - \cos\alpha]$.

9. When $\alpha = 32^\circ$, E for Paris is 0.1289 and E at the equator is 0.8480. Thus, the equator receives more than 6 times as much solar energy on a summer day than Paris receives on January 15, provided that the earth's tilt has changed.

14.R Review Exercises for Chapter 14

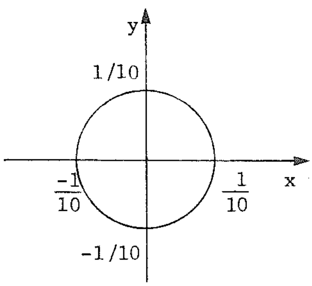
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



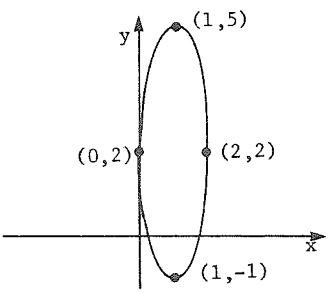
$4x^2 + 9y^2 = 36$ is the same as $x^2/9 + y^2/4 = 1$. This is an ellipse with intercepts at $x = \pm 3$ and $y = \pm 2$.

5.



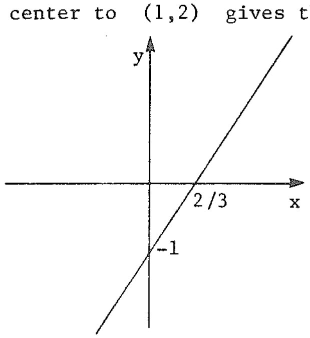
$100x^2 + 100y^2 = 1$ is equivalent to $x^2 + y^2 = 1/100$, which is a circle centered at the origin with radius $1/10$.

9.



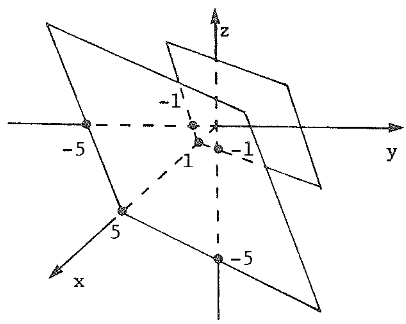
By completing the squares, we have $(9x^2 - 18x + 9) + (y^2 - 4y + 4) = 9 = 9(x - 1)^2 + (y - 2)^2$ or $(x - 1)^2 + (y - 2)^2/9 = 1$. This has the form $X^2 + Y^2/9 = 1$, which is an ellipse with intercepts at $X = \pm 1$ and $Y = \pm 3$. Shifting this ellipse's center to $(1,2)$ gives the desired graph.

13.



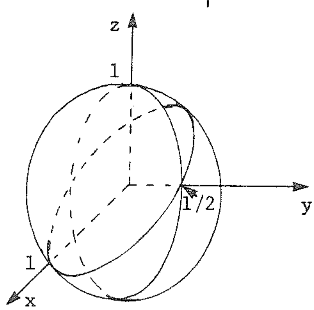
$c = 2 = 3x - 2y$ implies $2y = 3x - 2$ or $y = 3x/2 - 1$. This is a line with slope $3/2$ and y-intercept at -1.

17.



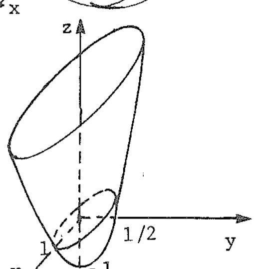
$c = x - y - z$ is a plane for any value of c. These planes are parallel, whose position depends on the values of c.

21.



This is an ellipsoid centered at the origin with intercepts $(\pm 1,0,0)$, $(0,\pm 1/2,0)$ and $(0,0,\pm 1)$.

25.

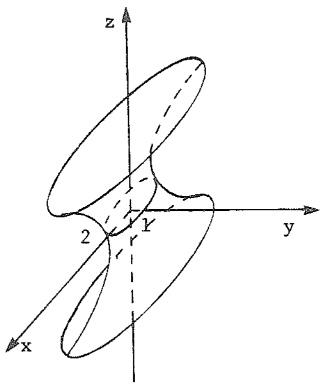


$x^2 + 4y^2 = 1 + z$ is an ellipse parallel to the xy-plane when z is held constant if $z > -1$. This is an elliptic paraboloid with intercepts $(\pm 1,0,0)$, $(0,\pm 1/2,0)$, and $(0,0,-1)$.

29. (a) $(x/a)^2 + (y/b)^2 = 1 + (z/c)^2$ is an ellipse in any plane parallel to the xy-plane when z is held constant.

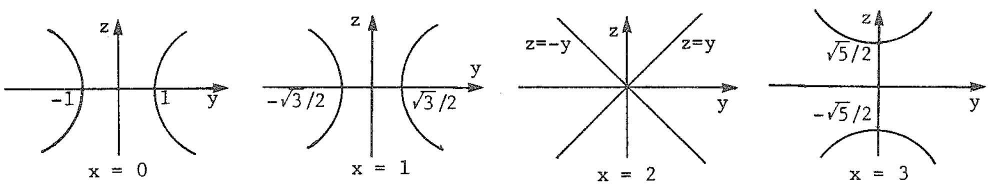
(b) When x is held constant, we have $(y/b)^2 - (z/c)^2 = 1 - (x/a)^2$, which is a hyperbola opening along the y-direction and parallel to the yz-plane if $|x| < a$. If $|x| = a$, then $(y/b)^2 = (z/c)^2$, or $y = \pm bz/c$, which is two straight lines. If $|x| > a$, then the hyperbola opens in the z-direction. A similar analysis may be done for constant y.

29. (c)

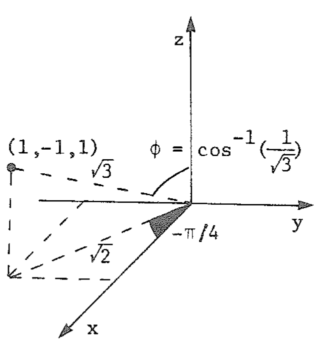


The easiest way to sketch this is to draw ellipses parallel to the xy-plane and enlarge them as $|z|$ increases.

In the yz-plane, $x = 0$ gives the equation $y^2 - z^2 = 1$ which is a hyperbola with y-intercepts, ± 1. For $x = 1$, the equation is $y^2 - z^2 = 3/4$, so the y-intercepts are $\pm\sqrt{3}/2$ for this hyperbola. When $x = 2$, $y^2 = z^2$ or $y = |z|$, which is two straight lines. And $x = 3$ yields the equation $y^2 - z^2 = -5/4$. This is a hyperbola with z-intercepts, $\pm\sqrt{5}/2$.



33.



To convert cylindrical coordinates to rectangular coordinates, use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. To convert from rectangular coordinates to cylindrical coordinates, use $r = \sqrt{x^2 + y^2}$,

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases},$$

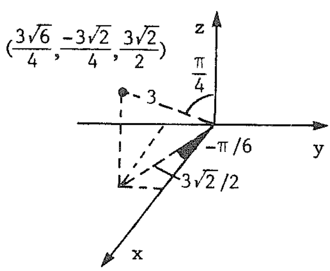
and $z = z$.

33. (continued)

The conversion from rectangular coordinates to spherical coordinates uses $\rho = \sqrt{x^2 + y^2 + z^2}$, $\phi = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2}) = \cos^{-1}(z/\rho)$,

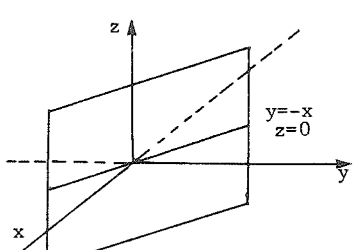
$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geqslant 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}$. Thus, $r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$;

$\theta = \tan^{-1}(-1/1) = -\pi/4$; $\rho = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$; $\phi = \cos^{-1}(1/\sqrt{3})$. Thus, the cylindrical coordinates are $(\sqrt{2},-\pi/4,1)$ and the spherical coordinates are $(\sqrt{3},-\pi/4,\cos^{-1}(1/\sqrt{3}))$.

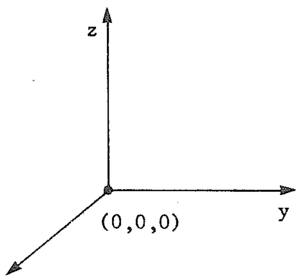
37. 

To convert spherical coordinates to rectangular coordinates, use $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$. After converting to rectangular coordinates, r is obtained from $\sqrt{x^2 + y^2}$. θ is the same as the spherical coordinate. Thus, $x = 3 \sin(\pi/4) \cos(-\pi/6) = 3(\sqrt{2}/2)(\sqrt{3}/2) = 3\sqrt{6}/4$; $y = 3 \sin(\pi/4) \sin(-\pi/6) = 3(\sqrt{2}/2)(-1/2) = -3\sqrt{2}/4$; $z = 3 \cos(\pi/4) = 3\sqrt{2}/2$; $r = [(3\sqrt{6}/4)^2 + (-3\sqrt{2}/4)^2]^{1/2} = 3\sqrt{2}/2$. Thus, the rectangular coordinates are $(3\sqrt{6}/4,-3\sqrt{2}/4,3\sqrt{2}/2)$ and the cylindrical coordinates are $(3\sqrt{2}/2,-\pi/6,3\sqrt{2}/2)$.

41. Replacing (ρ,θ,ϕ) by $(\rho,\theta + \pi,\phi + \pi/2)$ has the effect of moving each point 180° around the z-axis, followed by a 90° rotation from the positive z-axis.

45. 

$x + y - z = 0$ is the equation of a plane. Plot the points (0,0,0) , (1,0,1) and (0,1,1) . Then draw a plane through these points.

49. 

$-(x^2 + y^2/4)$ is nonpositive and z^2 is nonnegative; therefore, the only solution is $(0,0,0)$.

53. The equation of the tangent line is $(f(t_0),g(t_0),h(t_0)) + (t - t_0) \times (f'(t_0),g'(t_0),h'(t_0))$. In this case, $(f'(t),g'(t),h'(t)) = (3t^2, -e^{-t},-(\pi/2) \sin(\pi t/2))$, so the tangent line is $(2,1/e,0) + (t - 1) \times (3,-1/e,-\pi/2)$.

57. The velocity vector of $f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ is $f'(t)\underline{i} + g'(t)\underline{j} + h'(t)\underline{k}$ and the acceleration vector is $f''(t)\underline{i} + g''(t)\underline{j} + h''(t)\underline{k}$. By the sum rule, $\underline{\sigma}'(t) = \underline{v}(t) = \underline{\sigma}_1'(t) + \underline{\sigma}_2'(t) = [e^t + 2t/(1 + t^2)^2]\underline{i} + (\cos t + 1)\underline{j} - \sin t\ \underline{k}$; $\underline{\sigma}''(t) = \underline{a}(t) = \underline{\sigma}_1''(t) + \underline{\sigma}_2''(t) = [e^t + (2 - 6t^2)/(1 + t^2)^3]\underline{i} - \sin t\ \underline{j} - \cos t\ \underline{k}$.

61. The arc length is $L = \int_1^2 [(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]^{1/2} dt = \int_1^2 (1 + 1/t^2 + 2/t)^{1/2} dt = \int_1^2 [(t^2 + 1 + 2t)/t^2]^{1/2} dt = \int_1^2 [(t + 1)/t]\, dt = (t + \ln t)\big|_1^2 = 1 + \ln 2$.

65. (a) Let $\underline{r}(t) = a(t)\underline{i} + b(t)\underline{j} + c(t)\underline{k}$. $\underline{F} = m\underline{a} = m\underline{r}''(t) = -k\underline{r}(t)$, so rearrangement gives $\underline{r}''(t) = -(k/m)\underline{r}(t)$. Thus, the differential equations are $a''(t) = -(k/m)a(t)$, $b''(t) = -(k/m)b(t)$, and $c''(t) = -(k/m)c(t)$.

(b) From the solution of the spring equation, we have $a = a_0 \cos \omega t + (a_0'/\omega) \sin \omega t$, where $\omega = k/m$. $\underline{r}(0) = \underline{0}$ implies $a_0 = b_0 = c_0 = 0$ and $\underline{r}'(0) = 2\underline{j} + \underline{k}$ implies $a_0' = 0$, $b_0' = 2$, and $c_0' = 1$. Thus, $a(t) = 0$. Similarly, $b(t) = (2m/k) \sin (k/m)t$, and $c(t) = (m/k) \sin (k/m)t$.

69. Let the curve be parametrized by the vector $(x,y,z) = \underline{\sigma}(t) = (t,f(t),0)$. Then $\underline{\sigma}'(t) = \underline{v} = (1,f'(t),0)$ and $\underline{\sigma}''(t) = \underline{a} = (0,f''(t),0)$. Recall that curvature is given by the formula $\|\underline{v} \times \underline{a}\|/\|\underline{v}\|^3$. $\underline{v} \times \underline{a} = f''(t)\underline{k}$, so $\|\underline{v} \times \underline{a}\| = |f''(t)|$; $\|\underline{v}\|^3 = (1 + |f'(t)|^2)^{3/2}$, so the curvature is $|f''(t)(1 + |f'(t)|^2)^{-3/2}|$.

73. (a) The intersection of the surfaces is given by the equation $x^2 + 1 + z^2 = 3$ or $x^2 + z^2 = 2$, which is a circle of radius $\sqrt{2}$ centered at $(0,1,0)$ in the plane $y = 1$. One parametric form of the curve is $x = \sqrt{2}\cos t$, $y = 1$, $z = \sqrt{2}\sin t$. (There are other solutions to this exercise.)

(b) Using the answer we got in part (a), the velocity vector is $(-\sqrt{2}\sin t, 0, \sqrt{2}\cos t)$. The curve is at $(1,1,1)$ when $t_0 = \pi/4$, so the tangent line is $(1,1,1) + t(-1,0,1)$.

(c) The curve is self-intersecting, so restrict the interval to $0 \leqslant t \leqslant 2\pi$. The arc length is $\int_0^{2\pi}\sqrt{(-\sqrt{2}\sin t)^2 + (0)^2 + (\sqrt{2}\cos t)^2}\,dt$ $\int_0^{2\pi}\sqrt{2}\,dt = \sqrt{2}t\Big|_0^2 = 2\sqrt{2}\pi$.

TEST FOR CHAPTER 14

1. True or false.

(a) The derivative of $\underline{u} \times \underline{v} = \underline{u}' \times \underline{v} + \underline{v}' \times \underline{u}$.

(b) The graph of the unit sphere is the graph of a function of x and y .

(c) The derivative of $3\underline{i} + 2\underline{j} - \underline{k}$ is a scalar, namely 0 .

(d) If the spherical coordinate ϕ is $5\pi/8$, then the point (ρ,θ,ϕ) always lies below the xy-plane, provided $\rho > 0$.

(e) Any equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A , B , C , D , E , and F are constants, describes a conic section

2. A sphere of radius 2 is centered at $(-2,-1,0)$.

(a) Write the equation of the sphere in spherical coordinates.

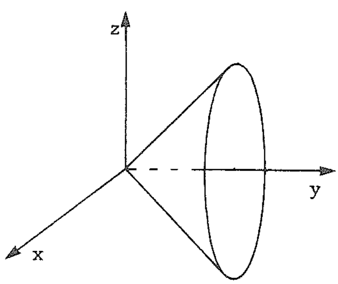
(b) What are the cylindrical coordinates of the sphere's center?

3. (a) Let $\underline{w}(t) = (t^5 + t)\underline{i} - (\sin t)\underline{j}$. What is the derivative of $\underline{w}(x^3 + 5x^2 - x + 1)$ with respect to x ?

(b) If $\underline{u}$, $\underline{v}$, and $\underline{r}$ are vector functions, find a formula for $(d/dt)[\underline{u}(t)\cdot\underline{r}(t)]\underline{v}(t)$.

4. (a) Sketch the graph of $2y = x$ in xyz-space.

(b) Using the definition of a cylinder, explain why the surface in part (a) is or is not a cylinder.

5. 

The surface shown at the left is a cone (not of the elliptical type).

(a) Write the general equation of the surface.

(b) Describe the level curves for y = constant.

6. (a) Sketch the parametric curve $(e^t,e^{2t},2e^t)$, $0 \leqslant t \leqslant 2$.

(b) Compute the length of the curve sketched in part (a).

(c) What is the velocity vector for the parametric curve?

7. Describe the graphs of the following:

(a) θ = constant in cylindrical coordinates.

(b) $\phi = \pi/4$, $\rho \leqslant 0$ in spherical coordinates.

(c) $\theta = 30°$, $\phi = \tan^{-1}2$ in spherical coordinates.

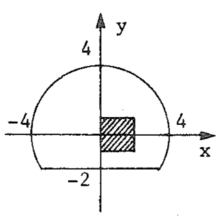
8. (a) Sketch the surface described by the equation $4x^2 + 4y^2 + z^2 = 1$.

(b) If the origin is shifted to $(-1,0,2)$, what is the equation of the new surface?

(c) If an insect crawls along the original surface from $(0,0,1)$ to $(1/4,1/4,\sqrt{11}/4)$ along the shortest path where $x = y$, how far does it travel? (Leave your answer in terms of an integral.)

9. (a) Compute the curvature of the circle $x^2 + y^2 = 9$.

(b) What is the principal normal vector to the circle at $(0,1)$ if the circle is traced out in a counterclockwise direction?

10. 

A certain city's boundary, as depicted, is given by the circle $x^2 + y^2 = 16$ for $y \geqslant 2$. The downtown area (shaded) is bounded by the lines $x = 0$, $x = 2$, $y = -1$, and $y = 1$. Panhandlers know that at (x,y), they can get $16 - x^2 - y^2$ dollars per day. However, in the downtown area, they expect to get $21 - x^2 - y^2$ dollars per day. Sketch a graph over the city's domain to show panhandlers how much they can expect to receive if they ask for handouts at (x,y).

ANSWERS TO CHAPTER TEST

1. (a) False; it is $\underline{u}' \times \underline{v} + \underline{u} \times \underline{v}'$.

(b) False; there are two values for z for most (x,y) where the graph exists.

(c) False; the derivative is the vector, $\underline{0}$.

(d) True

(e) False; only if the graph exists.

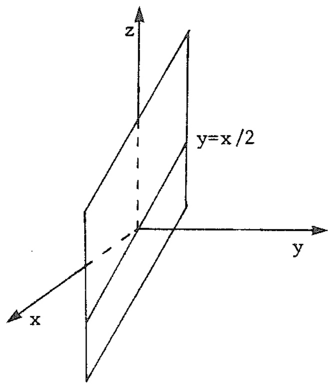
2. (a) $\rho^2 + \rho \sin \phi(4 \cos \theta + 2 \sin \theta) + 1 = 0$

(b) $(\sqrt{5},-5\pi/6,0)$

3. (a) $(3x^2 + 10x - 1)[(5(x^3 + 5x^2 - x + 1)^4 + 1)\underline{i} - (\cos(x^3 + 5x^2 - x + 1))\underline{j}]$

(b) $[\underline{u}'(t)\cdot\underline{r}(t) + \underline{u}(t)\cdot\underline{r}'(t)]\underline{v}(t) + [\underline{u}(t)\cdot\underline{r}(t)]\underline{v}'(t)$

4. (a)

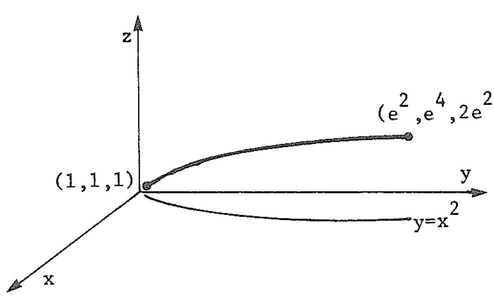


(b) The surface is a cylinder because z does not appear in the equation.

5. (a) $x^2 + z^2 = y$; $y \geqslant 0$

(b) The level curves are circles of radius $\sqrt{y}$, centered on the y-axis if $y \geqslant 0$.

6. (a)

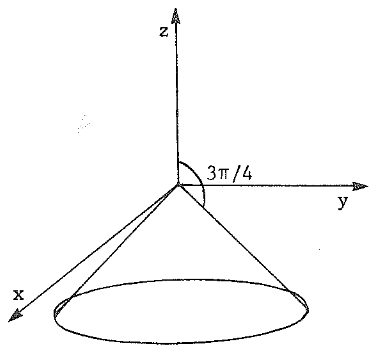


(b) $[2e^2\sqrt{4e^4 + 5} - 6 + 5 \ln (2e^2 + \sqrt{4e^4 + 5}) - 5 \ln 5]/4$

(c) $(e^t, 2e^{2t}, 2e^t)$

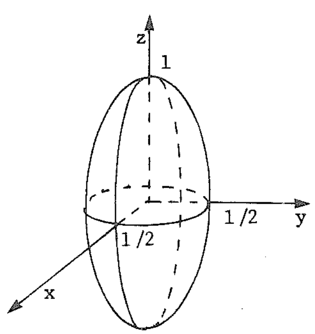
7. (a) xz-plane rotated θ radians.

(b) A cone lying below the xy-plane.



(c) A straight line.

8. (a)



(b) $(x + 1)^2 + 4y^2 + (z - 2)^2 = 1$

(c) $\int_{\sin^{-1}(\sqrt{11}/4)}^{\pi/2} \sqrt{\sin^2 t/5 + \cos^2 t}\, dt$

9. (a) 1/3

(b) $-\underline{i}$

10.